Inquiry in University Mathematics Teaching and Learning: The PLATINUM Project
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CHAPTER 16

Teaching Inquiry-Oriented Mathematics: Establishing Support for Novice Lecturers

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16.1. Introduction

The professional development of novice lecturers is considered to be of crucial importance in the PLATINUM project. In this chapter, we present what a group of lecturers[1] at Universidad Complutense de Madrid (UCM) (mathematicians and mathematics education researchers), constituted as a Community of Inquiry (CoI), considers valuable for the development of the professional knowledge of mathematics lecturers. With this aim, the group has designed learning tasks and developed seminars within the framework of the PLATINUM Project.

Referring to the three-layer model outlined in Chapter 2, this chapter describes the interaction between the second layer, *inquiry into teaching mathematics* (lecturers using inquiry to explore the design and implementation of tasks, problems, and activities in classrooms), and the third layer, *inquiry into research for a professional development programme for mathematics lecturers*, which takes the results in developing the teaching of mathematics in order to systematise advanced areas and professional development programs at the institutional level.

Focusing on the Inquiry-Based Mathematics Education (IBME) approach, the didactic design certainly has an essential role for the establishment of productive links between research and practice. However, for the didactic design to be effective, it must be considered not only as a by-product of research but also must be incorporated into the programs associated with the professional development of mathematics lecturers.

We present the methodological approach and several materials for professional development. We focus on the challenges and questions that appear in the design and execution of mathematical tasks for teaching matrix factorisation in the subject of numerical methods, both from a theoretical and practical point of view.

The rest of this chapter is structured as follows: Section 16.2 presents the UCM context. Section 16.3 is devoted to the UCM CoI and the fundamentals underlying the approach. Section 16.4 describes the design of materials for professional development in mathematics. Section 16.5 presents the *Matrix Factorisation* case, where the theoretical background is applied to the analysis of acquisition of knowledge and inquiry processes. Implementation results with novice lecturers are presented in Section 16.6. Finally, a discussion and some conclusions are included.

[1]We use the terminology of *lecturer* and *professor* as in UK English for university education. In Spain the equivalent term is *university teacher*. Also, in this paper, we refer to the novice lecturers enrolled in the course on Teacher Professional Development in Mathematics as *participants*, reserving the name *students* for the mathematics learners in the degree programmes experiencing the lessons.

16.2. Complutense University of Madrid

The Complutense University of Madrid, with nearly 72,000 students, is the largest university in Spain with two campuses in Madrid. It was founded in 1499 and is a centre of reference for Latin America. Its 26 faculties offer 320 official degrees (Bachelor's degrees, double degrees, Master's degrees, doctorates, and international degrees) and more than 300 continuing education degrees.

Given the multidisciplinary nature of mathematics, lecturers with very different backgrounds—including mathematicians, experts in mathematics education, engineers, and so on—teach different topics in the Mathematics Faculty or teach mathematics in other faculties.

UCM has a specific teacher training centre where the University Master's degree programme in Teacher Training for Secondary and Vocational Education and Language Teaching[2] is taught in collaboration with the Faculty of Mathematical Sciences for the specific area of Mathematics. This qualification is of an enabling nature to teach at these levels of education.

The professional development of lecturers is not linked to any specific qualification. UCM has a training plan for teaching and research staff,[3] although this is specially designed for lifelong education or continuing university teacher training, and not so much for initial training. Thus, the professional development for teaching of novice lecturers is developed through the faculties that offer studies in each specific subject.

Regarding teaching methodologies for novice lecturers, UCM has a programme for innovation in teaching (called *Innova Docencia*[4]), which encourages (voluntary) lecturers to try innovative teaching approaches and techniques.

Three bachelor's programmes are offered in the Faculty of Mathematical Sciences at UCM (Mathematics and Statistics, Mathematical Engineering, and Mathematics). In the curriculum of the latter there are subjects related to the teaching of mathematics, both at the secondary and university levels. In addition, the Faculty is the headquarter, since its foundation in 2007, of the Miguel de Guzmán Chair,[5] which has as objectives the analysis, research, and teaching of the reality, problems, and perspectives of mathematics education in Spain and internationally. Since its inception, it has promoted research in mathematics education through various research projects.[6]

In the Faculty of Mathematical Sciences at UCM some attempts have been made to connect mathematics education to inquiry-based mathematics education. Reference is often made to problem solving, in which there is a long tradition of research and practice in the field that goes back to the seminal work of György Pólya (1945, 1954). In Spain, Miguel de Guzmán, professor at UCM and ICMI president in the 90s, encouraged teaching and learning in this direction by publishing various books and developing a theoretical framework that gives an essential role to problem solving. The teacher training programmes are under this approach, promoted with the support of the Spanish Ministry of Education and with the collaboration of international experts such as Alan Schoenfeld (1985). More specifically, emphasis was put on reflections on mathematics methods, focusing on the development of mathematical competences and metacognitive skills that can be interpreted in terms of inquiry habits of minds and problem-solving attitudes (de Guzmán, 1995). From the inquiry perspective it is worth mentioning the international study carried out at UCM by Miguel de Guzmán

[2] `www.ucm.es/masterformacionprofesorado`

[3] `https://cfp.ucm.es/formacionprofesorado`

[4] `https://eprints.ucm.es/pid.html`

[5] `http://blogs.mat.ucm.es/catedramdeguzman`

[6] `http://blogs.mat.ucm.es/catedramdeguzman/proyectos-de-investigacion`

and other researchers (de Guzmán, 1998) on the difficulties of the transition from secondary school to university where some mathematical-didactical problem areas regarding epistemological, cognitive, and sociocultural aspects were identified.

The design of materials and resources for professional development of university lecturers of mathematics in the PLATINUM Project at UCM has had important precedents, under the problem-solving approach and based on the Design-Based Research Collective (2003). Since 2009, the research group of mathematicians and mathematics educators has designed and implemented different courses to provide university lecturers and research assistants with educational tools enabling them to better design, implement, and analyse teaching and learning processes (Corrales & Gómez-Chacón, 2011; Gómez-Chacón & Joglar-Prieto, 2010; Gómez-Chacón et al., 2020). The PLATINUM project represents further progress in terms of an international contrast with other IBME approaches and a wider dissemination of practices.

16.3. Community of Inquiry (CoI) at UCM

The starting point for the UCM CoI was in our opinion the collaboration in the ICMI and/or the Miguel de Guzmán Chair described in the precious section. In fact, most of the lecturers that finally joined the PLATINUM project collaborated via these organisations in innovation-related projects: the project leader was already in charge of courses for novice lecturers, and everyone tried to apply new teaching approaches in their lectures. Even though most of them still lacked the theoretical context related to inquiry-based learning, in most cases they tried to encourage students to explore the instructional materials, ask questions, and discuss proposals. Taking these ideas into account, the UCM CoI initially started with eight lecturers. Besides these eight members of the 'core' CoI we founded an extended CoI including several colleagues interested in improving their teaching and PhD students / novice lecturers who want to learn new teaching techniques. In particular, an extended CoI was formed thanks to seven members of *Proyecto Innova-214: ESCEMMAT-Univ1*[7] focused on professional development for novice lecturers in mathematics. These young lecturers participated in the professional development program.

The members of the 'core' UCM CoI learned about inquiry-based learning, trying to apply it to their lectures, collaborated to implement new lessons/assignments, and participated in the general PLATINUM meetings. All these activities were periodically discussed in local meetings, where the members of the CoI presented their ideas and results so that the rest of the members could give feedback. In these meetings general PLATINUM issues were also addressed. Although for most of the PLATINUM project these meetings were held offline, we changed to online meetings due to the COVID19 pandemic. This was not the only difficulty, because the pandemic also forced us to change the teaching mode, making the implementation of some activities more difficult.

In the UCM CoI, Inquiry-Based Mathematics Education (IBME) is considered widely as a way of teaching wherein students are invited to work in ways similar to how mathematicians work (cf., Dorier & Maaß, 2020). Within the PLATINUM project, the development of this perspective is based on the three-layer model presented in Chapter 2 of this book and in (Jaworski, 2019), and it is rooted in the idea of communities of inquiry.

[7] http://blogs.mat.ucm.es/catedramdeguzman/proyectos-de-investigacion/escemmat-univ

Through this CoI, a substantial growth of knowledge and awareness of inquiry-based approaches to teaching and learning mathematics has taken place. These approaches encourage students' engagement, creativity, and conceptual understandings of mathematics.

16.4. Designing Materials for Professional Development

The UCM-PLATINUM team's idea of *professional development* is based on an epistemology of professional knowledge which takes into account the contextualised nature of the teacher's experience (experience knowledge) and the personalised knowledge of the practice (Frade & Gómez-Chacón, 2009; Gómez-Chacón & De La Fuente, 2019). In essence, this 'competence of the actor in her/his context' recognises the subject as the main actor of his development and co-creator of process of her/his professional development.

Our proposals for the professional development aim to provide lecturers and research assistants with educational tools that enable them to better design, implement, and analyse teaching and learning processes in mathematics. In the PLATINUM project, we have prioritised in the lecturers' professional development the sense of belonging to a CoI and the design of mathematical tasks under the IBME approach. For the design of professional material, we have used as strategy the design methodology represented in Figure 16.1. It has been intentionally used to explore the interplay between the layers in the three layer-model presented in Chapter 2. Our conceptualisation of IBME takes explicitly into account this specific nature of mathematical inquiry and the essential contribution of internal inquiry to the development and structuring of mathematics as a domain of knowledge. Following inquiry-based learning epistemological bases, it is not only important to acquire new knowledge, but also to develop analytic and experimental strategies to reach new knowledge.

Figure 16.1 represents the design process for the materials to be implemented in professional development courses for novice lecturers. It is divided into four phases: *Discover*, *Define*, *Design*, and *Develop*. In the creation process of the tasks, possible ideas are defined. These ideas come from the needs identified and the teaching objectives necessary for the development of strategic knowledge in the 'learning to teach mathematics' at the university level. One of the key aspects is the confirmation of problem definition and the creation of a solution through design and class development.

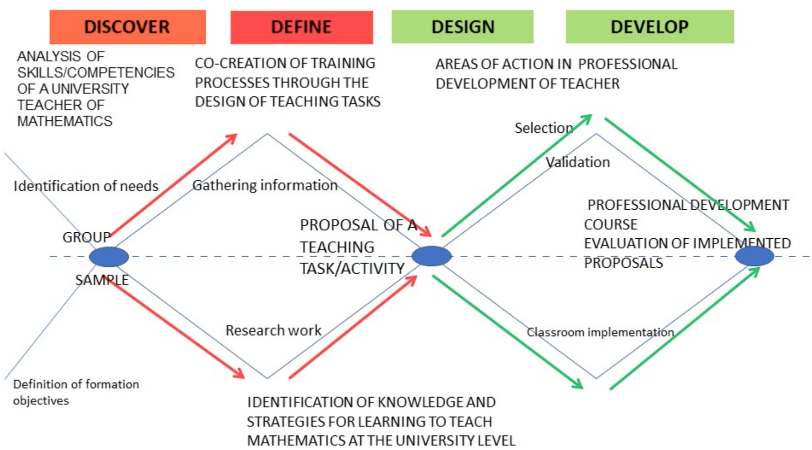

FIGURE 16.1. Design process of creating and evaluating instructional materials in a professionalisation context.

In the proposed conception, research and development are mutually involved. Professional development is viewed from a reflexive position concerning practice. The aim is for new teachers to join in a research project over the practice, in the sense of questioning and analysing it to understand it, and even transform it.

The diagram aims to map the divergent and convergent stages of the design process by showing designers' different ways of thinking. In our case the designer was a team of four PLATINUM members: mathematicians whose fields of research and expertise are applied mathematics, algebra, computer science, and mathematics education, respectively.

16.4.1. Discover. The first phase, *Discover*, marks the beginning of the design. This phase corresponds to a deep contextual dip into the challenge of professional development of novice lecturers at the Mathematics Faculty of UCM. It involved different working sessions of the CoI. These discussions were supported by a review of literature about research in algebra at university level, data from the narrative techniques applied to mathematicians, video recording of discussion group meeting of CoI, and a questionnaire about teaching algebra (for senior and novice lecturers). We used them to understand difficulties of students in algebra as well as to identify needs of professional development in order to integrate the IBME approach. Some of them that will be highlighted in the developed materials are:

(1) concepts of linear algebra—matrix diagonalisation and factorisation, applications such as rigid motions (Linear algebra courses occupy a dominant position in the overall undergraduate mathematics curriculum.);
(2) interconnections between linearalgebra and numerical analysis, linear algebra and geometry, linear algebra and rewriting logic;
(3) inquiry forms and task typologies according to the students from several Bachelor's programmes (Mathematics, Computer Science, Engineering).

The following excerpts from one of the professors and another novice lecturer teaching linear algebra illustrate these aspects in relation to professional development in inquiry approaches:

> My objectives within the PLATINUM project are fundamentally two. On the one hand, to improve my teaching work and, on the other hand, help, as far as possible, to improve the teaching practice of other colleagues. The topics that interest me the most have to do with algebra and especially with linear and matrix algebra: the relationship between concepts such as linear applications and their representation as matrices, the multiple ways of factoring a matrix and its applications, etc. Another issue of interest is how to approach these topics according to the type of student to whom the class is directed. Returning to the PLATINUM CoI, knowing what my colleagues do in other subjects allows me to verify that many of the concepts of linear and matrix algebra appear in different contexts, which I can take advantage of to motivate my students (Professor's narrative, April 2019).

> On the other hand, we work with a programming language, which has a particular syntax and semantics, so we need to go beyond theory and implement the mathematical ideas. Right now, students memorise 'patterns' so they can apply the same structures to similar problems, but they do not really understand why one structure is more appropriate in a given situation, so minor changes in the specification make students fail in their assignments. I think inquiry might be appropriate to take both Mathematics and Computer Science together. If they reason and interact with the concepts, they can relate both fields effectively and reach a deeper understanding. I expect discussions to take place for both theoretical and practical issues. Moreover, the experience in this

Master's course can be replicated, if successful, to Bachelor courses in Logic (Novice lecturer's narrative, April 2019).

16.4.2. Define. The second phase, *Define*, represents the definition phase, i.e., the moment when insights are defined and refined. This phase aims to identify patterns and to reach conclusions based on collected data in the previous phase. The main activities held during the *Define* phase are:

- defining the didactic-mathematical problems to be answered through the design of classroom-directed activities (tasks, units, teaching actions, etc.);
- identifying the approach under which to develop the IBME project;
- selecting the study group, courses, subjects, and levels to carry out;
- anticipating the professional skills to be acquired by novice lecturers.

Table 16.1 shows the main inquiry projects that were designed and implemented in the classroom with undergraduate students from different Bachelor's programmes at the Mathematics Faculty (IO3). Some of them were selected to be considered as base materials to be used in professional development courses. In order to define them, priority was given to the teaching of concepts, taking into account the interdisciplinary nature of knowledge, and how an inquiry approach can help in their understanding. We describe them briefly.

Teaching linear algebra and video games. Affine transformations and rigid motions are the main concepts. An open inquiry project designed for undergraduate students in the Bachelor's programme Videogame Development. It consists of a progressive approach to linear algebra according to three pedagogical principles: the *principle of*

Subject / Field	**Topic**	**Level**	**Connections**	**IBME model/inquiry features**
Algebra	Diagonalisation	Bachelor	Numerical linear algebra, matrix computations, dynamical systems	Flipped classroom methods
	Rewriting logic	Master	Technology Maude language	Semi-guided inquiry
Algebra and Geometry	Affine transformations and rigid motions	Bachelor Master	Technology Interactive geometry system	Open guided inquiry Video-game project
	Isometries and tessellations of the Euclidean plane	Bachelor	Technology Interactive geometry system	Semi-guided inquiry
Numerical analysis	Matrix factorisation	Bachelor	Algebra	Semi-guided inquiry
	Numerical methods for ODEs	Bachelor	Calculus	Semi-guided inquiry
Calculus	Functions, derivation, differentiability	Bachelor		Semi-guided inquiry
	Rolle's Theorem	Bachelor		Semi-guided inquiry
	Elements of Ordinary Differential Equations	Bachelor		'Escape Room' project

Table 16.1. Examples of inquiry projects.

concretisation, *principle of necessity* and the *principle of generalisation and formalisation* (Harel, 1989). Specific techniques and resources for the teaching of the concepts of related applications and rigid motions are proposed (see Chapter 7).

Diagonalisation of matrices. This project seeks to relate the concepts of diagonalisation, eigenvalues and eigenvectors to the matrix of a linear mapping with respect to a base formed by eigenvectors, if it exists, using simple examples. In professional development, the aim is to encourage reflection on teaching practice, focusing on the involvement of students in learning concepts such as diagonalisation, so that they are able to guess if certain properties are fulfilled or discard them if they find a counterexample. Using the 'flipped classroom' concept, video, work guides, and evaluation instruments are prepared on the topic.

Specification in rewriting logic. This project aims to formalise the specification of complex software systems through rewriting logic. This formalisation will allow students to prove properties on these systems using different techniques using logic, which requires students to abstract the details and focus on the mathematical properties underlying these systems. Because this project is designed for students in computer science, we design the units using games, interaction between the lecturers and the students, and inquiry.

Matrix factorisation. This project tries to establish connections between numerical analysis and linear algebra. Matrix factorisations are methods for reducing a matrix to a product of simpler matrices, so that complex matrix operations can be simplified by performing them on the decomposed matrix rather than on the original matrix itself. They are fundamental not only in linear algebra for solving systems of linear equations but also have many applications (see Section 16.5).

16.4.3. Design and Develop. The third phase, *Design*, seeks to generate ideas and prototypes for professional development courses. The fourth phase, *Develop*, focuses on the adjustments and further refinements that must be performed to produce more mature prototypes in the medium and long term and offer them in professional development courses for mathematics novice lecturers. The final aim is to produce professional development models and examples of associated practices for inducting these lecturers into inquiry-based mathematics education. So, one of the main activities and goals during this phase is testing, adjusting, and validating the prototype as materials for lecturers' professional development. Some of them were used in the workshop or seminar during the academic course 2019-2020.

Taking into account this implementation, one of the main goals in the *Develop* phase was performing brainstorms with the CoI and end users, defining the essence of the given ideas for teaching using an inquiry approach, comparing them to the core of the mathematical problems and professional competences for novice lecturers.

We would like to emphasise that the model should not be understood as a one-way flow (Figure 16.1). In fact, the designers of the tasks navigated through phases; they intensified or abandoned the use of tools and techniques, and moved back and forth as the challenge and the feed-back with the real context progressed (testing, adjusting, and validating).

16.5. The Matrix Factorisation Inquiry Project

The purpose of these materials for professional development is to support lecturers in developing and extending their range of practices in the subject of numerical methods through an IBME approach, which is taught to students of various study programmes related to mathematics.

Matrix factorisation was chosen as the most appropriate topic for applying inquiry-based learning. In the implementation in the classroom, the students moved between the experimental, theoretical, and algorithmic levels. More specifically, the students began factoring a matrix and had to generalise the procedure to obtain an algorithm and also conjecture a theoretical result. The results of the implementation and the good feeling of the students, as revealed in the questionnaires (Figure 16.3 and Figure 16.4), shows the adequacy of inquiry in this teaching process.

16.5.1. Professional Competences to Develop by Novice Lecturers. The competences to be developed through the collaborative working sessions are

- to learn how to design and elaborate materials on subject matter of the course program that will
  - allow the students to consolidate the concepts addressed in previous courses and topics. In particular, LU-factorisation, making the LU (or $PA = LU$) factorisation of a matrix understood as a natural consequence of Gaussian elimination used to solve a system of linear equations; and
  - encourage the development of mathematical intuition and different representations of the same concept; and
- to encourage reflection on teaching practice, emphasising the involvement and detection of students' difficulties in learning and managing of matrix factorisations that allow them to efficiently solve systems of linear equations, using direct methods.

16.5.2. Teaching-Learning Tasks. The inquiry-based task about matrix factorisation for the students (a documented teaching unit on the PLATINUM website) was prepared by two mathematics professors with wide experience in teaching at university level and, in particular, in the subject of numerical methods. In the design of these materials in professional development of lecturers, other members of the PLATINUM project—the authors of this chapter—participated, too.

In the professional development of novice lecturers we are of opinion that the fundamental issue to be studied is not merely how to present materials better, rather, it is ultimately how students learn and perceive concepts in numerical analysis in connection with linear algebra (previous knowledge in the undergraduate).

Using an inquiry-based approach, we are not proposing another way of teaching numerical methods; rather, we propose to provide a systematic framework for lecturers to become a better learning facilitator, to ask thought-provoking questions, to design lessons that facilitate conceptual understanding of key concepts in numerical methods, to help students make mental constructions of mathematical objects, and to create a lasting effect in student learning of mathematics in general.

16.5.3. Design Process. In what follows we present the design of the materials. We also describe how the experience of the senior professors is presented to novice lecturers, breaking it down into the most significant phases. In addition, novice lecturers are presented with different tasks to make them go through the process of choosing mathematical notions, tasks to be developed with the students, and their timing. According to the process explained in Section 16.4 (Figure 16.1) we describe these phases for the LU-factorisation task.

Discovery phase. We start with the determination of mathematical concepts that will be the focus of the tasks, as well as the discussion of difficulties that novice lecturers identify in the subject of numerical methods.

We focus on the process of developing materials for second year students in various mathematics-intensive Bachelor's programmes (Mathematics, Mathematical Engineering, and Mathematics and Statistics, as well as the dual programme Mathematics and Statistics and Economics). The course is called *Numerical Methods* and includes several topics related to matrix computations. At UCM, it is taught four hours per week during a four-month period, two of them are theoretical, one hour is dedicated to problem solving and one more hour is dedicated to practice in the computer lab where MATLAB[8] is used. It is within the hours of theory and problems that the activity carried out by the team is introduced.

In general, students in the Bachelor's programmes in Mathematics show interest and are motivated by their studies. However, they often have difficulty managing matrices and using them appropriately to solve problems. In particular, there are difficulties in understanding matrix factorisation. The results are felt as abstract, surprising, and unnatural. For these reasons, the factorisations are not introduced as algebraic equations proved by induction on the size of matrix. Instead, factorisation is developed from Gaussian elimination, a technique well-known by the students. Nevertheless, this way of presenting factorisation does not avoid some difficulties in its use and, especially, in its implementation.

Analysis and definition phase. The novice lecturers analysed the different options and approaches according to what was obtained in the exploratory phase and determined what the materials are prepared on (Definition phase in Figure 16.1). The topic of matrix factorisation is chosen, a concept that is often seen by students as unnatural, not at all intuitive, and difficult to implement.

The theme is framed within the methods of numerical resolution of systems of linear equations. The objective of LU-factorisation is to create two triangular matrices—lower triangular matrix L, upper triangular matrix U—such that $A = LU$. This allows to easily solve the given system of equations $Ax = b$, computing the solution of the triangular systems $Lw = b$ and $Ux = w$. The matrix L is obtained by placing on the main diagonal the numbers 1 and the Gaussian elimination multipliers in the places indicated by their indices. The matrix U is obtained as the matrix resulting from the elimination process. Eventually, it may be necessary to exchange two rows to continue the Gaussian elimination process and so instead of the factorisation $A = LU$ we have $PA = LU$, where P is the permutation matrix that accounts for those exchanges.

Materials design phase. Within the framework of inquiry-based teaching, the analysis of the material that the lecturers of the PLATINUM project have implemented is crucial (Design phase in Figure 16.1). With these examples of tasks, the novice lecturers were invited to identify tools to guide their students, and how to encourage reflection on the inquiry process and the acquisition of mathematical concepts. Different tasks are proposed:

Task 1: Reflect on how to approach in a more natural way the matrix factorisations studied ($A = LU$ and $PA = LU$), linking them to Gaussian elimination.

Task 2: Find examples of matrices that are versatile enough to exemplify both types of factorisations.

Task 3: Design of the exercises and their scheduling, so that they can serve to provoke a gradual sequence of conjectures and their eventual refutations or confirmations. Thus, a matrix that admits LU-factorisation is considered, and students are directed to apply the Gaussian method to it, storing the multipliers in L and calculating the product of L by U to reach the original matrix and provoke a first conjecture. After

[8] www.mathworks.com

that, a matrix is chosen as a result of exchanging the rows of the previous one and the calculations are reproduced, seeing that the original matrix is not obtained, which will lead the students to revise their guess. Finally, we start from the second matrix and repeat the calculations but store the multipliers in the place that would occupy the zeros. In this way, the product of the matrix L thus stored by the matrix U is a permutation of the starting one, which leads the students to guess a new result. Once the exercises and questions for this activity have been decided, a worksheet is prepared to be distributed among the different groups of students.

The inquiry nature of the tasks is discussed, focusing on the characteristics of an inquiry-based mathematical task regarding the following dimensions:

(1) How and who initiates the questioning? (*Questioning*).
(2) Which is the nature of the problem? (*Nature of the problem*; *Mathematical knowledge*).
(3) To what extent are students responsible for the inquiry? (*Student responsibility*).
(4) How the lecturer's goals are made explicit? (*Goals*).

These dimensions portray the modalities of inquiry-based teaching and learning that range from teacher- to student-centred. In the Matrix Factorisation project, as it carried out at UCM, the lecturers elaborate the questioning after considering students' concerns. The tasks are formulated as a partially open-ended problem: Students have to cope with an open task and limited material already prepared. It is the design of tasks that breaks with the usual routine of classes in which the lecturer introduces the concepts to perform, later, practices or problems where these concepts appear. In this project, autonomous study is encouraged, as well as the ability to work in groups, to ask oneself questions, and to guess results. In the classroom, students have to perform a series of tasks that test their knowledge and ability to manipulate matrices. In addition, the proposed tasks include questions in which students are encouraged to conjecture results, confirm or refuse them with new examples and adjust their conjectures in view of the conclusions thus obtained (*Inquiry*).

In the problem classes the students were divided into groups of four, trying to encourage both group work and discussion among equals. The teacher supervised and resolved the doubts that the different groups posed, always trying to encourage students to work independently. In short, the aim was for the student to participate actively rather than being just a mere receiver of information. Students are asked to justify their conclusions with respect to knowledge or evidence. Figure 16.2 presents some examples proposed by the lecturers.

Finally, Task 4 is proposed to prepare a survey to ask the students to make explicit what they have learned during the inquiry session.

Task 4: Prepare a sample survey for students to complete after the activity.

After the lecturers participating in the sessions had shared their ideas, they were shown the two questionnaires proposed in experience with undergraduates. Questionnaire 1 (Figure 16.3) was posed to the students after finishing the tasks. A few days later, when the subject corresponding to this task had already been explained, Questionnaire 2 (in Figure 16.4) was presented to the students.

Development phase. The sequence and results of the implementation with mathematics undergraduate students of the Bachelor's programme at the Mathematics Faculty are presented to the group of participants in the professional development workshop. The steps followed in this session are listed below.

First part

We consider a matrix A to which the Gauss method can be applied without permutations. In fact, as we will see, the element that is left on the diagonal at each step (which is called the *pivot*) is the largest among all those that are, in the corresponding column, from the diagonal to the end of the column.

```
A=[8 4 2 0;4 6 1 1; 2 1 4.5 1;0 1 1 2.25]
```

```
A = 4x4
    8.0000    4.0000    2.0000         0
    4.0000    6.0000    1.0000    1.0000
    2.0000    1.0000    4.5000    1.0000
         0    1.0000    1.0000    2.2500
```

We are going to apply the Gauss method to matrix A, **always** choosing the linear combination of rows in which each row is subtracted from the pivot row multiplied by a number (which is called *multiplier*).

We save the initial matrix A in a matrix U in which we are going to make the transformations.

```
U=A
```

```
U = 4x4
    8.0000    4.0000    2.0000         0
    4.0000    6.0000    1.0000    1.0000
    2.0000    1.0000    4.5000    1.0000
         0    1.0000    1.0000    2.2500
```

For the first column, the pivot is 8. Therefore, we will have to subtract from the 2nd, 3rd and 4th rows the 1st row multiplied, respectively, by $\frac{4}{8}=0.5$, $\frac{2}{8}=0.25$ and $\frac{0}{8}=0$. We are going to store these numbers in the matrix L which, initially, is the identity matrix. We will save them in their **first column**, since they are the multipliers used to make zeros in the **first column** of U. We keep the multiplier used to make a zero in the **second row** of U (that is, 0.5) in the **second row** of L, that of the **third** (0.25) in the **third row** of L and that of the **fourth** (0) in the **fourth row** of L.

```
L=eye(4);  % Initialize L to identity matrix
L(2,1)=0.5; L(3,1)=0.25; L(4,1)=0;
L
```

```
L = 4x4
    1.0000         0         0         0
    0.5000    1.0000         0         0
    0.2500         0    1.0000         0
         0         0         0    1.0000
```

What does matrix U look like after having made these linear combinations of rows on it?

U=

FIGURE 16.2. Task example.

Questionnaire 1

(1) How was the experience of facing an open problem, in which you are asked to conjecture from concrete examples?
(2) What aspects did you find less clear, more difficult to understand or realise? Has the use of 'clues' been useful?
(3) How much do you value the sharing of the conclusions?
(4) Have you benefited from the discussion? Has it been useful to you? What for?
(5) What do you think about this way of approaching the results? Do you think it would be positive to include more activities of this nature in the course? What activities can you think of?
(6) Rate the activity globally from 1 to 10, with 10 being the maximum score.
(7) Write below any comments you want to make about the task.

FIGURE 16.3. Questionnaire 1.

Questionnaire 2

(1) Do you think the task has facilitated your understanding of Gaussian elimination en LU factorisation of a matrix?
(2) Please indicate the aspects that you think have been easiest for you (if any).
(3) Now that you know the theory and implementation, would you change anything in the design of the task?

FIGURE 16.4. Questionnaire 2.

(1) Introduction about the objectives, creation of groups of four students, and delivery of worksheets.
(2) Accompaniment of the lecturer. The lecturer supervised the groups during the activity. Problems with comprehending some statements came to light and were clarified. Some clarifications were also needed about what the students were expected to do.
(3) Processes of conjecture and justification in the investigation. It is proposed to conjecture on different aspects (possibility of LU-factorisation and how to obtain it). We worked on the different examples of matrices chosen in Task 2. The greatest difficulty lies in the students' lack of habit inferring general results from concrete facts. For example, this is the answer of one of the student' groups to the first question of Questionnaire 1, which can be considered representative of all of them: "I think it is very useful because you force yourself to ask yourself and try to discover things that you would not normally think about if they were explained directly to you". Another answer to this question was: "It is difficult but when working in a group, ideas that had not occurred to me came up and it was easier to reach the conclusion", which shows that students positively value the contribution of working in a group.
(4) Once the task was finished, a discussion group was organised. First, a representative from each group communicated the group's answer to each of the three questions of the worksheet to the other groups, so that all students would be aware of the conjectures of all their peers. Next, through a debate led by the teacher, the groups assessed, assumed, or criticised the responses and opinions of the other groups. To this end, the lecturer encouraged the members of each group to explain their arguments. Often, these were reformulated taking into account the answers given by the other groups. To this end, the lecturer encouraged the members of each group to explain their arguments. There were different situations of the type "We had thought... but hearing what group X responded we realised that it was not so."
(5) Results of the experience. A synthesis of the answers from the student groups was presented to the novice lecturers, as well as the type of feedback given to the students, where the difficulties encountered were reflected, the mistakes made in the development of the practice were analysed, and guidance on mathematical concepts and processes worked on was given. Some arguments were surprising. For example, one group pointed out the symmetry of the matrix of the first example, an irrelevant detail in this context.

    However, most of the answers were quite reasonable. For example, here we have some typical answers to the first question of the worksheet (about a matrix A which has LU-factorisation with no permutations): "In this 1st part, we come to the conclusion that the product of the transformer matrix L by the upper triangular matrix U gives rise to the original matrix A," "Every matrix A can be decomposed as a product of an upper triangular matrix U and another lower triangular matrix L, such that $A = LU$, where L 'stores' the transformations by rows to triangularise A." And the corresponding answers to Question 2 of the worksheet (about a matrix B which has no LU-factorisation without exchanges of rows): "We suspect that the change in rows 1, 3, and 4 has caused that the only row that remains the same is 2 and therefore the LU product is different from the initial matrix B," "Clarify that the transformations that 'stores' L are not only multipliers, all the elementary operations that are carried out in U have to be reflected in L."

(6) Results of the survey applied to undergraduate students on the evaluation of the proposal. In general, the activity was valued positively by the undergraduate students with an average score of 8.44 points out of 10. There was a general perception that, at the beginning, what was being pursued was not well understood. With the teamwork and the clarifications made by the lecturer, the difficulties and the paralysis that arose from having to guess a result were overcome. Teamwork also contributed to losing the fear of writing something wrong. The pooling of ideas from the different groups was also highly valued; it was thought that it brings out aspects not considered in the first stage (when working in groups), as well as difficulties different from one's own.

The results of the questionnaire showed quite clearly the usefulness of the activity for a better approach to matrix factorisation. Some of the answers to Question 3 of Questionnaire 1 were of the type "Positively because by exchanging ideas you self-correct or reaffirm yourself on your own and everything is much clearer" or "Very positive because having thought about the exercise before and having your own conclusions, sharing helps to identify your own mistakes and, also, see other points of view on how to interpret the exercise." In the same vein the answer of another group to Question 4 of Questionnaire 1 was: "Yes, seeing the way of understanding the exercise by the other groups and the different contributions is useful to better understand the activity. Further, you get to ask yourself doubts that had not arisen before".

The results of the survey show quite clearly the usefulness of the activity for a better approach to matrix factorisation. We present some answers:

> In general, everything is easier with practice, since it is more visual than if the method is taught in a theoretical and conventional way (Student).

> I think the method used is very didactic, since you do not learn by seeing, but by doing; and, sincerely, in mathematics many times opportunities of this style are missed, to be able to approach the problem directly, to pose it with the help of some clues, and to obtain conclusions by yourself, or in this case in a group. I find it really interesting, and the best way to teach mathematics; to force you to think from time to time, instead of giving it all thought (Student).

> I think it's pretty good, mainly the part about stating a theorem and seeing how we were wrong (Student).

> The sharing of ideas with my peers has made me learn new ways to approach a problem as soon as I see it (Student).

The results of the implementation encouraged novice lecturer's reflection on teaching practice, emphasising the involvement and detection of students' difficulties in learning and managing of matrix factorisations that allow them to efficiently solve systems of linear equations via direct methods. Several challenges are formulated in relation to the awareness of mathematical knowledge in the inquiry process, particularly in the transition of processes experimentation, theorisation, and algorithmisation.

16.6. Implementation Results With Novice Lecturers

Valorising professional development sessions, we tried to answer the following question: what characterises the attitude adopted by novice lecturers regarding the IBME approach? Our analysis revealed three areas in which novice lecturers adopted a critical attitude: towards mathematics; towards learning mathematics; and towards teaching mathematics. These areas were found to align with the stages depicted in Figure 16.1— *discover*, *identify*, *design*, and *develop*—and relate to the learning, practice, and teaching of mathematics. In the analysis of the results we have taken into account three conditions: (1) awareness that the lecturers' experiences may have been different, (2) reflection on that experience, and (3) judgement of the aspects to be included in future teaching improvements.

16.6.1. Critical Attitude Towards Mathematics. In adopting a critical attitude towards mathematics in an inquiry-based learning approach two aspects arose: (1) the understanding of the mathematical content to be taught and (2) the nature of student participation.

In the notion of understanding, 'the capacity for abstraction and complex reasoning,' which entails the understanding of theoretical concepts in action and the updating of those that serve as support for the acquisition of new ones, is required. In relation to the subject of numerical methods, a component of an algorithmic nature was highlighted and characterises mathematics as a sequential content. The contributions on the algorithmic nature of mathematics were of interest:

> The first and most important difficulty students have in Numerical Methods is with the concept of algorithm. In particular, retrieving and using in an orderly and sequential manner the elements necessary for any of the methods studied. This translates into the second difficulty, which is to produce code that implements that algorithm. Students do not, in general, find it easy to break a task down into its elementary parts that can then be coded (Novice lecturer – Identify phase).

These observations provoked a discussion in the CoI about the integration of mathematics as content and process. Processes emphasise the dynamic nature of mathematics, how it is created and how it evolves over time. In this case, the understanding of the concept of algorithm and the mechanisms underlying the process of algorithmisation was highlighted.

Regarding the second aspect, the nature of student participation in undergraduate lessons, two views were made explicit. One was by senior lecturers and professors who stressed that students when they participated in Numerical Methods classes under the inquiry approach, engaged with the tasks and developed shared knowledge. It highlights that students' experience of mathematical meaning and connections can emerge in different ways. There may be different reifications (representations) of the same concept or they may relate to the way in which the same reification can lead to different experiences (tasks and activities) in which learners are involved.

However, as a second view, the participants highlighted more the "lack of curiosity to go into deep and abstract concepts." One of the participants seemed to recognise this when he said:

> The lack of curiosity, understanding this as the pupils' natural response to theoretical concepts and their lack of interest in finding out more than what they have seen. It is therefore the teacher's responsibility to sequence.

Mathematics is reified through classroom tasks and discussions, through procedures, representations and transformations, patterns, relations and connections, theorems and proofs. All of this involves precision in language and the development of skills and competences on the part of the teacher in designing tasks from this point of view. Less awareness of ways of developing meaning through participation is evident in novice lecturers.

16.6.2. Critical Attitude on Learning Mathematics. In this section, we consider the ways in which participants adopted a critical stance toward learning mathematics through inquiry. Two aspects stand out in the discussion because of the contrast between the perspective of the professors or seniors lecturers and novice lecturers: learning is fundamentally experiential and social and learning is a matter of student engagement.

Firstly, learning is fundamentally experiential and social on the part of the participants; learning is more individual and in relation to the subject and mathematical knowledge. In the discussion of the results presented on collaborative work as a means of advancing knowledge, an emerging attitude appeared among novice lecturers: awareness of the social construction of mathematical knowledge. In their own words:

> Something to bear in mind, too, is teamwork. After the discussion and considering our own experience, working in a team enriches the students, as they support each other and exchange information and ideas. This helps to better understand and develop concepts. So, it would be good to develop teamwork strategies for activities and that it can be something transversal (and dynamic, i.e. not always with the same team) where there is a forum even in the whole class to discuss and contribute. Collective learning can be very beneficial, especially with abstract concepts and the complexity they bring with them (Novice lecturer).

Two themes were identified in the discussion: meaningful learning and learning as an active process. There is an evolution of learning towards something more experiential, social, and engaged. In the training session we paid explicit attention to the students' learning process as indicative of an (emerging) critical attitude at university level, given the dominant tendency of lectures. By the end of the training session, the shift from a conceptual understanding linked to content towards a reflection on the more holistic learning process becomes explicit. Students contrasted different ideas about learning, and/or offered critiques of their approaches to learning. We consider that in novice lecturers, variation in learning experience is essential to reflect on, contrast, and criticise one's own past and current experiences.

16.6.3. Critical Attitude on Mathematical Meaning and Processes. Several participants indicated that their practice with engineering students focused on the procedural part and on the theoretical-practical connection, which they found to be an essential deficit in the students. Learning in this context turns into memorisation and the application of previously memorised procedures. However, despite this finding, the novice lecturer in charge found it difficult to design tasks that were open-ended and left students a wide margin for exploration. Figure 16.5 shows some of the items in the sequence proposed in response to Task 3 by one participant (Design of problems and their pacing, so that they can serve to provoke a gradual sequence of conjectures and their eventual refutation or confirmation) (see Section 16.5.3).

In Figure 16.5 one can see how closed and guided the inquiry is. The participation in the CoI and the joint discussion helped to raise awareness of these aspects and to see what questions were left for student understanding and creation.

However, in the beginning the lecturers indicated a shift towards understanding, interpretation, and creativity as important parts of inquiry approaches, where meaningful learning of mathematics is crucial. As evidence pointing to this evolution in

(1) Let A be the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix}$. Obtain its echelon form by Gaussian elimination. The transform the above procedure by multiplying with elementary matrices. Do the same with the matrix $B = \begin{pmatrix} 2 & 6 & 1 \\ 2 & 7 & 4 \\ 1 & 0 & 1 \end{pmatrix}$.

(2) Let A and B be the matrices of the previous exercise. Using the above, get the LU factorisation of both matrices.

(3) Let $\left\{ \begin{array}{rcl} x_1 + 2x_2 & = & 1 \\ x_1 + 3x_2 + 3x_3 & = & 1 \\ x_3 & = & 1 \end{array} \right.$ and $\left\{ \begin{array}{rcl} 2y_1 + 6y_2 + y_3 & = & 2 \\ 2y_1 + 7y_2 + 4y_3 & = & 1 \\ y_1 + y_3 & = & 0 \end{array} \right.$

(a) Write both systems in the form $AX = b$ and $BY = b'$.

(b) Obtain the factorisation $A = LU$ and $B = L'U'$ of the coefficients matrices of both systems.

(c) Solve the systems $LZ = b$ and $L'Z' = b'$.

(d) Using the above solutions, solve the systems $UX = Z$ and $U'Y = Z'$.

(4) Let C be the matrix $\begin{pmatrix} 0 & 2 \\ 2 & 6 \end{pmatrix}$.

(a) Compute the reduced echelon form of the matrix C by Gaussian elimination.

(b) Repeat the process using elementary matrices.

(c) Is is possible to obtain a factorisation $C = LU$? If the answer is affirmative, compute it, otherwise explain why.

(d) The same question with $PC = LU$ for some permutation matrix P.

(5) Compute the factorisation $D = LU$, where $D = \begin{pmatrix} 3 & 1 & 7 \\ 5 & 6 & 4 \\ 4 & 2 & 1 \end{pmatrix}$.

FIGURE 16.5. Items of Task 3 by one novice lecturer.

thinking, most of the participants, in the beginning, did not consider matrix factorisation to be a difficulty in Numerical Methods and that meaningful work had to be done on this concept in the classroom. When asked about this, they were inclined to give answers such as: :

> I have not singled out matrix factoring as a difficulty, because when we talk about difficulties, we tend to focus more on theoretical, deep and abstract concepts; but it is true that many basic matrix processes tend to fail... I consider that the essential mathematical concepts for this subject are: elementary row operations, elementary matrices, Gaussian elimination and systems of linear equations (question posed in the Definition phase).

At the end of the evaluation session, this same participant indicated:

> After the final discussion and after seeing the implementation in the classroom, I have had doubts and, above all, concerns. I have been able to see that when it comes to setting exercises (tasks), I give quite a lot of guidelines and the activity is quite closed. This can mean that the student's creative capacity is not developed and that he or she does not ask questions to be able to continue investigating. Therefore, as a teacher, I should try to guide the student in the exercises, leaving him or her to reach a conclusion that can then be verified or rejected.
>
> To do this, I have to work on the transition from theory to practice. I feel that I am able to do simple enough steps supported by examples when explaining the theory,

> but when it comes to the student working on his/her own, I am not able to reflect this. I could see that in the proposed exercises there is a lack of freedom to develop ideas, which makes the learner too constrained. Now I am beginning to consider that open questions or reflections by the students will help them to develop that mathematical capacity for argumentation and meaningful conclusion.

We see a reflection on the learning process among the participants and an assessment of the value of inquiry in going deep into 'mathematical meaning' and into the mathematical processes involved in the concept. This opens up for teacher training processes the deepening of the nature of tasks under the inquiry approach where the dimensions of questioning (key questions to move towards the solution), mathematical knowledge, responsibility of learners in the investigation, objectives, etc., are all important.

16.7. Concluding Remarks and Ongoing Work

The case study portrays our experience of development of novice lecturers' professional knowledge and practice regarding inquiry-based teaching. We focused on the methodological strategy that we used to establish operative connections between the processes of inquiry at two levels: *Inquiry in teaching mathematics* (lecturers using inquiry to explore the design and implementation of tasks, problems and activity in classrooms) and *Inquiry in the design of professional development programme* for mathematics lecturers.

The proposal to support novice lecturers in the design by sequencing in four phases—*Discover*, *Define*, *Design*, and *Develop*—has been effective. It has fostered the crucial characteristics of inquiry-based teaching: the origin of questioning, the nature of the mathematical problem, students' responsibility in conducting the inquiry, the management of student diversity, and the explanation of the teacher's goals.

Some challenges in order to support professional development of novice lecturers were raised: what do we mean by an inquiry-based task in mathematics? What is specific to inquiry in mathematical work and differentiates it from another knowledge? In the proposal about matrix factorisation, the conceptualisation of IBME took explicitly into consideration the specific nature of mathematical inquiry and the essential contribution of internal inquiry to the development and structuring of mathematics concepts through experimentation, theorisation, and algorithmisation. These actions contributed to the lecturers' knowledge and competences, but also to the formation of habits of mind for inquiry.

Finally, for us, the ideas raised in this project made us search for a strong interplay between research and professional development activities in relation to IBME. We also want to understand better how communities of inquiry among mathematics lecturers at university level can be established, maintained, and extended via the professional development of novice lecturers.

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