Inquiry in University Mathematics Teaching and Learning: The PLATINUM Project
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https://doi.org/10.5817/CZ.MUNI.M210-9983-2021-15

CHAPTER 15

Two Decades of Inquiry-Based Developmental Activity in University Mathematics

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15.1. Introduction

The authors of this chapter work at a Mathematics Education Centre (MEC) at Loughborough University (LU) in the UK. We teach mathematics and mathematics education and we do research in mathematics education. This case study discusses research and development activities in which the MEC has been engaged for over 15 years including inquiry-based activity which is now related to the PLATINUM project. Our learning from these has been important for our individual development in both research and teaching.

Our teaching activities have involved collaborations between colleagues in the MEC, the Department of Mathematical Sciences (DMS) and the Foundation Studies Programme (FSP). During these 15 years, we have worked within a university culture of mathematics teaching and learning influenced by both the national milieu and the local policies of our university itself. In particular, we can point to

(1) the impact of a culture of university education that permeates practices in the UK. For example, issues related to school education and the preparation of students for university study;
(2) three government-initiated Research Excellence Frameworks[1] (REFs) over the 15 year period assessing research across all departments of all universities. This impacts the amount of Government support flowing into the university;
(3) at LU, the reorganisation from Faculties to Schools; the ways of organising lectures and tutorials; the domination of research over teaching; recent moves to make teaching development more important (e.g. the Teaching Excellence Framework, or TEF[2]).

Mathematics teaching at university level in the UK and beyond has followed a traditional path for many years with the main elements comprising large cohort lectures together with some forms of tutoring (Alsina, 2001; Pritchard, 2010). LU has largely followed this pattern. Towards the end of the millennium, university mathematics teachers became aware that students entering university seemed no longer well qualified for the content and pedagogy of university mathematics (Hawkes & Savage, 2000). This was largely attributed to changes to the curriculum in schools, where, for example, mathematical proof was not required. This raised issues about the kinds of (extra) provision that could be needed. For example, many universities introduced some form of 'bridging course:' at LU, a one-year Foundation Studies Programme was

[1] www.ref.ac.uk
[2] www.gov.uk/government/collections/teaching-excellence-framework

introduced in which students developed their skills and understanding in mathematics and science courses in preparation for entry to a bachelor degree programme. The demands of the REF have resulted in confirmation of a point of view that, in universities, research is of higher importance than teaching. In recognition of this position, the UK government instituted the TEF which also now assesses teaching quality. This has resulted in more focus on the nature and quality of teaching in universities.

15.2. Chapter Structure

15.3. History: we set the scene, positioning the growth of our inquiry-based activity in the context of both national educational and local university initiatives and structures, and influencing political, social and educational perspectives.
15.4. The Teaching Group: a Community of Inquiry focusing on a group of mathematics and mathematics education teachers working together to influence teaching and its development.
15.5. Inquiry-based Tasks in a Foundation Mathematics Course: discussing a developmental research project embracing teacher-researchers, student-partners and post-graduate students in a community of inquiry to develop tasks and teaching units for Foundation Studies students.
15.6. Teaching Engineering Students—a developmental research approach: the development of inquiry-based teaching of engineering students; where successful and where not successful.
15.7. Discussion: focusing on our learning as set out in the sections above and its relations to activity in the PLATINUM project.

15.3. History

The Mathematics Education Centre (MEC) at Loughborough University was created in 2002, within the School of Mathematics, and comprised a drop-in centre for mathematics support (The Mathematics Learning Support Centre, MLSC) plus responsibility for service teaching (including science, engineering and economics). In 2007, the MEC became a research centre focused on research into the teaching and learning of mathematics at university level and has diversified more recently to include all levels of mathematics education. The link between the MEC and the DMS is very strong where teaching is concerned (all staff contribute to the teaching of mathematics or statistics) and, in recent years, with introduction of the TEF, more emphasis has been placed on student learning and the development of teaching. However, research in mathematics education is very different from research in mathematics and, for REF purposes, they make a return to different assessment panels. Several initiatives have been undertaken to involve mathematicians with research into developments in learning and teaching. One initiative, the seminar series "How we Teach", was overtly focused on developing teaching (details follow below).

Parallel influences on research and teaching encouraged the MEC to study the development of teaching. Developmental research, often inquiry-based, became one feature of research in the MEC and included studies which pioneered inquiry-based approaches: for example, inquiry into students' use of digital proofs (Alcock & Wilkinson, 2011; Roy et al., 2017), inquiry into the teaching of linear algebra (Jaworski et al., 2009; Thomas, 2012), an innovation in teaching to promote engineering students' more conceptual understanding of mathematics (ESUM—details to follow below). These aspects of the history of the MEC are important as forerunners of inquiry-based research and development in the PLATINUM project: in particular, the three-layer model of inquiry-based practice has its roots in this work together with related research at the University of Agder, Norway. The first author has a long history in inquiry-based developmental research and has influenced the conceptualisation of inquiry in PLATINUM. This was built on developmental research in both the UK and Norway taking place at school level. A key element of inquiry-based learning and teaching at school level was the idea of forming inquiry communities among practitioners, teachers and didacticians. An inquiry community was seen as a group of practitioners who shared inquiry-based approaches to teaching and learning and supported each other in their development (Jaworski, 2008). At university level, a parallel is to form such inquiry groups between mathematicians and mathematics educators. With this in mind, a series of seminars, with the title "How we Teach," was introduced in which one teacher (mathematician or mathematics educator) gave a short talk about their thinking in some aspect of their teaching. The aim was to generate a discussion of teaching amongst colleagues and thus to encourage everyone to learn from the discussion and to develop teaching. The seminars became a regular feature in the MEC (from 2009–2014); they led to warm relationships between those attending and an enhanced awareness of teaching approaches in mathematics. They were pre-cursors of a specially convened "teaching group" to promote developmental inquiry in mathematics teaching and learning and, subsequently to the centrality of "communities of inquiry" in PLATINUM.

In the three sections which follow, we present aspects of our inquiry-based activity which have been important for us and, we believe, important as examples of key processes and theoretical perspectives in the PLATINUM project. In the first, we discuss the Teaching Group, mentioned above. This can be seen as a community of inquiry where we explored or inquired into new approaches to teaching and learning in mathematics. The second is a research project (called Catalyst) in which we worked with former students to design mathematical tasks for their more recent peers. These tasks used digital software and were inquiry based. In the third, we refer to a research project (ESUM—Engineering Students Understanding Mathematics) in which inquiry based tasks and teaching approach were used to improve students mathematical understanding. A reflection follows to address reasons for why these approaches seemed not to be possible when working with another group of engineering students.

15.4. The Teaching Group: A Community of Inquiry

The Teaching Group at LU started its meeting in 2016 and fulfilled the need felt by several colleagues both in the Department of Mathematical Sciences and the Mathematics Education Centre at LU for a forum to meet and discuss teaching mathematics and statistics at university level. This forum was to facilitate meetings and discussion for academics with complementary expertise and teaching experiences so that a Community of Inquiry (CoI—see Chapter 2 for details) could be established. The model of the CoI fitted well the aims of the Teaching Group: participants wanted to share practice and learn from each other and from educational research about the problems and issues they encounter in teaching. One of the contextual reasons why such a forum was initially successful is that, in the UK, training for new lecturers is generally not discipline specific therefore new colleagues joining the department felt that they needed a forum to discuss the teaching of mathematics specifically. Before this Teaching Group took shape, mathematicians and mathematics educators had shared the seminar series called "How We Teach." These were a regular feature in the MEC (from 2009–2014) and brought together educators and mathematicians interested in mathematics learning and teaching. In each seminar one member of staff talked about their teaching and others joined in discussion exploring practices and issues (Jaworski & Matthews, 2011). The Teaching Group similarly included mathematicians, statisticians, mathematics educators, all in the School of Science at LU but it did not consist of seminar presentations and question and answer sessions. Rather the Teaching Group was an informal forum for colleagues to meet and propose topics for discussion connected to teaching and reflect on their own teaching and on the experiences of others. The group met every two months or so for three years. Membership of the group was fluid—with both new lecturers and more experienced staff joining at various times during the group's existence. We followed a community of Inquiry (CoI) model (see Chapter 2 of this book) in the sense that we:

Inquired: we made use of materials such as education books and research papers, we produced teaching material, and we reflected on our own practice. Much of the focus of the sessions came from issues we encountered in our own practice such as formative assessment for university mathematics and the use of guided notes when lecturing. When discussing summative assessment (e.g. the type of questions to introduce in exam papers for engineers) we explored and learned about the use of inquiry-based mathematical tasks with students. We thought about 'inquiry based' tasks as tasks where the students were not asked to perform a procedure in the questions, but were asked perhaps to analyse and investigate a scenario presented to them;

Learned: we inquired into our own teaching through learning about the educational research on teaching mathematics at university level, reflecting on how our own experiences were mirrored or otherwise in the research we read. We also reflected on each other's experiences and discussed what each of us could learn from them. Our learning was helped by the exchange of ideas and resources: educational research discussions and teaching practice experiences were considered together to enrich our understanding of the teaching of mathematics at university level.

Our meetings usually lasted two hours plus lunch time to carry on talking. Topics we covered included assessment (we talked and read a lot about summative assessment of mathematics at university and especially about exam question content and format), feedback to students, student attendance, types of formative questions, computer aided assessment. Each of us was tasked, before the meeting, to read and present a research paper to the group on the topic chosen and the conversation to follow explored what we could take, in our own teaching, from that piece of research. At other times one of us presented something that they did in their own teaching—a mode of giving students feedback, or a format for guided notes—and the conversation that followed revolved around others' ideas on how that item could be suitable or beneficial for their own practice. The predominance of assessment in our meetings was probably due to a contextual factor related to the UK and a local factor related to the institution we all belonged to. The national UK factor is that in the survey of university students' satisfaction that the UK government issues at the end of each academic year (the National Student Survey,[3] NSS) 'assessment and feedback' are the topics which consistently score the least satisfaction across institutions. Therefore, there is great emphasis across universities to discuss assessment and feedback. Together with this external factor our colleagues expressed a general dissatisfaction with how mathematics for non specialists (e.g., engineers) is assessed. Many of the mathematicians and mathematics educators at LU teach mathematics to engineering students, therefore there was a real interest in discussing assessment for non-mathematicians. The dissatisfaction our colleagues felt consisted of the doubts they held that the current exam

[3] www.thestudentsurvey.com

paper format assessed predominately procedural mathematics while—as it transpired from our meetings—they valued conceptual understanding of mathematics above procedural understanding. Therefore we set off to find questions that could be asked in the exam papers which could assess some of the conceptual understanding valued—and those questions, as in the example reported in Figure 15.1, may be inquiry questions. In order to do so we read literature about assessment, about factors facilitating assessment change for staff and students, and discussed examples of questions that could elicit more conceptual understanding.

For the differential equation $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 5e^x$

(1) Find the complementary function.
(2) Find the particular integral.
(3) Find the particular solution with the initial conditions: $y(0) = 2$ and $y'(0) = 0$.
(4) Is there anything special about the solution method if the right hand side of the differential equation was $5e^{-x}$? Give a short explanation (maximum of 2 sentences).

FIGURE 15.1. Example of questions in an exam paper for first year engineering students.

The last item in this question was an attempt on the part of the lecturer to include some tasks that were less procedural in the assessment of their engineering module (see also Chapter 6 in this book where inquiry based tasks are discussed). The lecturer reasoned that asking the students to investigate something ("Is there anything special ...") instead of asking the student to implement a procedure ("Find ...") may stimulate students to reflect on the mathematical situation rather than carry out a well-rehearsed procedure. Since then, some questions which are more open ended—and arguably inspired by inquiry based learning—have made their way in this and other assessment.

In April 2019 we stopped meeting due to the increasing time pressures on staff at the School of Science (one of the characteristics of a culture of university education that permeates practices in the UK). Around this period there was much staff turnover in the Department of Mathematical Sciences and it proved very hard to have new colleagues joining the group. The existing group members found the demands of their day to day tasks too high and the time for meeting informally with colleagues—albeit to learn about teaching—disappeared. For colleagues who were still on probation, research outputs had to be prioritised over teaching activities reflecting, when professional progression is considered, the predominant role, in UK universities, of research over teaching, as mentioned above. During the last session we acknowledged that the experience had been positive, and we all expressed the wish to resume the meetings after a break. However, the COVID-19 pandemic has meant that finding time was even more difficult and to this day we have not resumed the activities of this group.

15.5. Inquiry-based Tasks in a Foundation Mathematics Course

In this section we report on a research project (called Catalyst[4]) where three researchers worked collaboratively with students, using digital tools, to design inquiry-based mathematical tasks for the mathematics course of the Foundation Studies Programme (sometimes referred to as 'Level 0' or 'Year 0' of the university degree) at LU. This group formed a Community of Inquiry (CoI)—bringing together different

[4]HEFCE (Higher Education Funding Council for England) Catalyst Fund: Innovations in learning and teaching, and addressing barriers to student success A: Small-scale, 'experimental' innovation in learning and teaching. Project code: PK20.

perspectives and expertise. The mathematical focus of our activities was matrices and complex numbers.

Our CoI consisted of three mathematics education researchers (all experienced mathematics teachers, one being the teacher of the foundation mathematics course), four first-year engineering and science students (our 'Student Partners', SPs) who had taken the foundation mathematics course in the previous year and two post-graduate students who assisted with data collection and analysis. In this CoI, we met regularly to discuss progress in the project and to create a co-operative environment where the student partners[5] could feel empowered to share their views.

A pre-requisite to our work was the inclusion of the dynamic geometry software Autograph[6] whose designer introduced us to the software. Our first task was to decide on the topics around which we would create inquiry tasks. One of the education researchers favoured the inclusion of complex numbers in order to use the software to help students understand complex numbers conceptually (details of tasks and their use with students can be found in Chapter 6). The second topic chosen was matrices and their relationship with linear equation systems. Our aim was to explore these topics with our student partners to create inquiry-based tasks for use in the teaching of future foundation students, using the computer software to facilitate inquiry. The student partners were included throughout the design process: they learned to use the software and created the Autograph files that were used in regular teaching of Foundation students a few months later.

Our group meetings were lively events that created a relaxed environment where the student partners could feel free to contribute. Discussions centred around the mathematics of complex numbers and matrices, how they are taught in textbooks and in the foundation course, how else they could be taught, desirable characteristics of a task, how to utilise the software to formulate and present the tasks and what the effect could be on learning. For example, reflecting on our discussions of potential tasks, one student partner noted how his mathematical understanding changed. He wrote,

> Working with the Catalyst project team helped me in understanding the concepts of complex numbers and matrices at a much higher level as the whole team brainstormed and everyone talking about their methods and approach to the same task and seeing the difference between how a lecturer thinks and how a student thinks really gave a good insight into these topics. (SP Reflective narrative, 11 September 2018)

Thus, at one level of engagement in our CoI, we were located in the inner layer of the PLATINUM Theoretical Framework (see Figure 2.1 in Chapter 2) where we inquired together into mathematics.

For the teachers, this often overlapped with issues of teaching and learning of mathematics, the middle layer of the Theoretical Framework, especially when we discussed designing the tasks using the software. The student partners expressed this overlap when reflecting on their participation in the project:

> Initially just from playing around with the different functions on the software, then as we practised we saw more things we could do and it snowballed from there.
> It was almost like 'reverse-engineering' the questions, we would start with a normal tutorial question, see what the answer looked like on Autograph and then re-design the question with the visual cue providing the information as opposed to it being stated directly in words. (SP Reflective narrative, 14 November 2018)

and

[5]See Jaworski et al. (2018) for details of the nature of the collaboration.
[6]`www.chartwellyorke.com/autograph/index.html`

> This is what made me realise that using the graphs on Autograph could help people to see what they were trying to solve in order to understand how to solve it. (SP Reflective narrative, 18 November 2018)

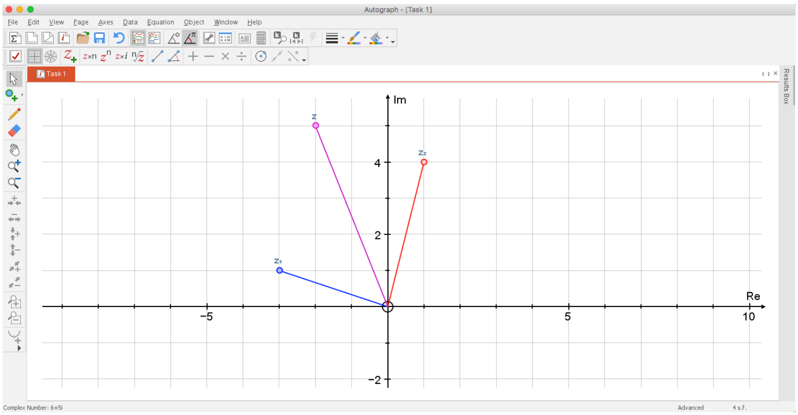
and

> When I think about what I learned throughout the Catalyst project, I cannot help but compare it to the way I originally studied the module during my own foundation year. In the case of complex numbers, I simply needed to understand what the symbols meant and how to manipulate them in a few specific circumstances. This was sufficient to answer the [exam or problem sheet] questions but it quickly became clear that designing them would require a much deeper understanding. (SP Reflective narrative, 11 September 2018)

As this was a research project, the education researchers, together with the two post-graduate students and sometimes the student partners, gathered data (audio-recordings of meetings, narratives and reflections, draft examples of the tasks) and analysed these using mainly qualitative methods, with several publications emerging (e.g., Jaworski et al., 2018; Treffert-Thomas et al., 2019). Thus, at this level of our engagement in our CoI we were located at the outer layer of the PLATINUM Theoretical Framework (Chapter 2, Figure 2.1) where we inquired explicitly into the inquiry aspects of our project with the aim of learning from our engagement and feeding back to inform practice.

As a result of several cycles of activity—designing, discussing, modifying tasks—we agreed on 6 complex number tasks and 5 matrices tasks. The tasks differed in nature but all had a dynamical element, making use of Autograph to either verify a result or explore a relationship further. When designing the tasks one of the student partners commented on the design process as 'reverse-engineering' (see citation above), meaning giving the answer and asking where it came from rather than asking "What is $a + b$?", the latter being a straightforward question with only a correct or wrong answer and not leaving any scope for investigation (an important observation in relation to PLATINUM IO3, see Chapter 6). Once confident in the use of Autograph, the student partners developed some tasks that pleased the teacher of the foundation mathematics course. The student partners formulated questions and produced Autograph files to go with the questions. The Autograph files were used (unaltered) in teaching and the questions were expanded collaboratively by the education researchers to create more context and guidance for foundation students. In addition, the questions (but not the Autograph files) were modified after use in the classroom following reflections and analyses by the education researchers. We found that students sometimes struggled with the wording of questions, in particular with the first (and perhaps easiest) task on addition of complex numbers (Figure 15.2).

Students did not focus on the geometric representation of addition of two of the complex numbers, i.e. the parallelogram (or triangle) law. Students instead decomposed complex numbers into their real and imaginary parts and verified their answers by adding these separately—in essence mirroring addition of vectors. With this task (and Task 2 on subtraction) we noted students' strong adherence to the conventions used in their foundation physics course including reference to the "resulting vector." The following year an adaptation to the terminology—from 'relationship' to 'mathematical relationship' adding also 'how are they connected'—did not produce a different result, students still decomposed into real and imaginary parts and often required a prompt in order to consider the geometric relationship. This led the teacher of the mathematics course to question the nature of the task and consider how to re-design



Task 1

There are three complex numbers labelled z_1, z_2 and z.
z_1 is to be kept fixed while z_2 and z can be moved.
Select z_2 and move it until z reaches the position $6 + 5j$.

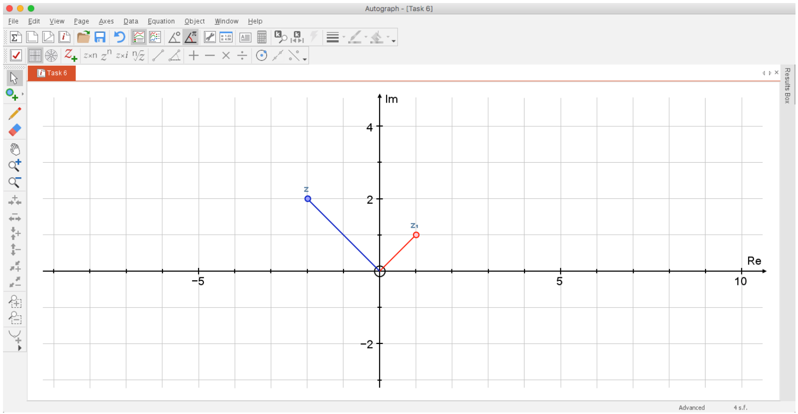
(a) What complex number is z_2? Right click and "Unhide All" to check your answer. The correct answer appears in green.
(b) What is the relationship between z_1, z_2 and z ?
(c) Now calculate by hand: With $z_1 = -3 + j$ and $z = 6 + 5j$, find z_2 such that $z_1 + z_2 = z$.
(d) Re-load *Task 1*. Move z_2 around the screen and notice how z changes as a consequence. What is the geometric connection between z_2, z and the complex number z_1 (which has stayed the same during your movements)?

FIGURE 15.2. AUTOGRAPH file Task 1: Addition of complex numbers.

in order that students engaged in the way that it had been envisaged. This is an example of teaching development (see Chapter 7). On the other hand, it seems that the contextual factors outweighed all others and hence addressing the contexts in which the tasks are delivered is an important consideration for anyone wanting to include IBME tasks in their curriculum. This is an example where research outcomes from the project led the teacher to inquire into her own teaching and make changes to how the tasks were presented in subsequent years. Thus the teacher's inquiry was located in the middle layer of the PLATINUM Theoretical Framework which links the inner and the outer layer (Chapter 2, Figure 2.1).

Multiplication by a complex number $r\angle\theta$ results in a rotation through an angle θ and an expansion of the plane by a factor r. Conceptually very powerful, the teacher's hope had been for students to experience this by working through the AUTOGRAPH tasks on multiplication. For example, Tasks 5 and 6 focused on the squaring and cubing of a complex number, respectively. In addition, Task 6 (Figure 15.3) had the option of a polar grid to visualise the cubing of a complex number, making it easier to 'see' the rotation and expansion.

As this was the last task out of the six, we found that students often did not have time to complete it. In a subsequent year the teacher of the course decided to label each task with a name such as Thelma, Abigail, etc., and laid worksheets out on a table so that students picked a task at random. In many ways that made some tasks more difficult. For example, the task on subtraction usually followed the task on



Task 6

There are two complex numbers labelled z_1 and z_2.

(a) Select z_1 and move it to different positions. There is a relationship between z_1 and z_2 but it is harder to see—so first move z_1 so that z_1 is real. What do you notice about z_2? Try different places for z_1 keeping it always a real number. When does z_2 have a larger modulus than z_1? When does it have a smaller modulus? When do they both have the same modulus? Remember to also try negative values for z_1.

(b) Try to find a relationship between the modulus of z_1 and the modulus of z_2.

(c) Click on the polar co-ordinate icon on the toolbar. Now allow z_1 to take any value, not only just real. Move z_1 and focus on the angle that it makes with the positive real axis. Also focus on the angle that z_2 makes with the positive real axis. Try to find a relationship between the angles as you move z_1 around.

(d) What do you think is the arithmetical relationship between z_1 and z_2?

FIGURE 15.3. AUTOGRAPH file Task 6: Cubing a complex number.

addition and students were able to pinpoint the relationship while when disjoint, they could not. As an experiment, the teacher will try to present tasks in pairs in the next iteration. Here again, the teacher's inquiry into the issues surrounding the teaching and learning of her students was located in the middle layer of the PLATINUM Theoretical Framework (Chapter 2, Figure 2.1).

This project profoundly affected the student partners whose (mathematical) understanding of complex numbers and matrices was greatly enhanced by participating in the design of the tasks. As one student partner wrote:

> It was only through designing the questions that I truly began to recognise and understand the relationships between complex numbers and Argand Diagrams. I believe this is because we went through the process of experimenting with different plots and observing how one change led to another, as opposed to reading and practising specific examples. It occurred to me just how much this process had influenced my understanding of the topics when I came across complex numbers in one of my third-year modules. During a lecture, it was immediately clear to me why the solutions appeared in conjugate pairs whereas many students had to spend some time revising the principle. (SP Reflective narrative, 11 September 2018)

Another student partner who designed matrices tasks also noted:

> I never understood . . . until I started doing the project. I thought perhaps other students might be going through what I went through when I was struggling with matrices. This is what made me realise that using the graphs on AUTOGRAPH could help people to see what they were trying to solve in order to understand how to solve it. (SP Reflective narrative, 18 November 2018)

All participants in the project enjoyed working as part of a community of inquiry. The education researchers were happy with the efforts of the student partners in designing the tasks. The student partners learned a lot—about mathematics and about designing tasks for use in the teaching and learning of mathematics. The teacher of the mathematics course acknowledged at one point that the tasks would probably not have come about had it been left entirely to the efforts of the teacher. However, the teacher also expressed some disappointment. Coming to PLATINUM after exposure to IBME activities and thinking deeply about IBME, the teacher of the course wrote after a project meeting:

> In retrospect, and when compared with other tasks that were presented [at the PLATINUM meeting] alongside mine, I began to think whether [the AUTOGRAPH tasks] were more 'hands-on' and 'explorative' than 'inquiry'. I always thought of them as tasks that could raise important conceptual understanding. I had thought less about how much time students would spend on 'inquiring'. I feel quite strongly that it is very hard to devise really good inquiry-based tasks. (Teacher Reflective narrative, 5 June 2020)

In the teacher's view 'inquiry' should involve a period of time reflecting on the different ways of going about finding a solution to the problem given. The AUTOGRAPH tasks were rather prescriptive, certainly allowing for exploration within the AUTOGRAPH environment but ultimately leading to a single (teacher approved) solution.

The PLATINUM project provided an opportunity to see a variety of different IBME tasks raising our own understanding of their potential and scope. Many of us in the PLATINUM project were teaching mathematics at university level but contexts (degree in mathematics, engineering, teacher education, etc.) and level (first year, second year, post-graduate, etc.) as well as topic area (calculus, complex analysis, modelling, etc.) differed greatly. Just as we discussed tasks in our local CoI, the wider discussion in the PLATINUM CoI inspired us to question the goal of presenting a task and what students may do to solve it. The challenge now is to incorporate aspects of the tasks we have seen, shared and discussed in the PLATINUM CoI into new or our own mathematical contexts.

15.6. Teaching Engineering Students

As mentioned above, one of the tasks of the MEC was to lead the teaching of engineering students in mathematics courses. Several members of the MEC were very experienced in this work and had contributed to the writing of the HELM books.[7] Supported by the Dean of the Engineering Faculty, a team of three teachers from the MEC (an inquiry group, CoI) decided to design a teaching/learning innovation: an inquiry-based approach to teaching a mathematics module for a cohort of engineering students using inquiry-based tasks. All three contributed to the design of the project (Engineering Students Understanding Mathematics, ESUM) and one was the lecturer

[7]HELM—Helping Engineers Learn Mathematics—is a set of around 50 workbooks presenting key ideas in a range of mathematics topics. They were produced at Loughborough University by members of the Mathematics Education Centre for use by students in the Engineering departments. They have been widely used in and beyond their original focus at LU and in other UK universities. They are freely available from `mec@lboro.ac.uk`.

for the cohort. Funding was received from the national HE-STEM programme and it paid for a fourth member of the team to act as researcher in the project, observing teaching, collecting data and aiding reflection.

The design of the teaching involved inquiry-based tasks for small group work in tutorials and the use, by the lecturer, of more open questions in lectures, seeking to engage students in participation in both types of session. Use of small group activity in tutorials was part of the innovation. Groups were assessed on a small project tackling inquiry-based tasks. A great deal was learned from the various stages of the project which fed back into the teaching of two successive cohorts. Several publications charted our learning in this project (e.g., Jaworski & Matthews, 2011; Jaworski et al., 2012).

In the style of ESUM, it would have been extremely valuable to repeat this inquiry activity in the teaching of other cohorts of engineering students. For one cohort in particular, the lecturer in their mathematics module was the same teacher as in the ESUM project. Unfortunately, she did not have the support of an inquiry group, or funding for a researcher to collect data etc. However, she hoped it might be possible to use some of the tasks from ESUM and to build some inquiry-based ideas into the teaching.

When a new lecturer was appointed to teach a module for a particular cohort of students, it was common, in their first year at least, to follow the specification of the module material and use the same teaching plan and assessment tools as in the previous teaching. This she did, with the only change being the replacement of 'in-class-tests' with a digital version, using STACK software[8] and the inclusion of some inquiry-based tasks in (otherwise traditional) tutorials. The STACK tests supported an inquiry approach to mathematical questions, providing feedback for students. Otherwise, lectures were conducted in a fairly traditional way following the previous structure of the course.

The STACK tests proved very popular and were used again with a new cohort. However, the lecturer was very disappointed that she had not found it possible to make the module more inquiry based. We present an account of her teaching of the module, with extracts from her own personal reflections.

> Here I am addressing inquiry in the second layer in our PLATINUM model: 'inquiry in teaching mathematics'. This means that I am reflecting on my own teaching, recognising my goals for teaching, the issues that arise in relation to these goals, and ways in which teaching might be developed or improved.

In the previous semester, she had taught a module on introductory mathematics to a cohort of 200 students in the department of Aeronautical and Automotive Engineering. These students had been recruited with a wide range of mathematical experience: some had high level qualifications (grade $A^\star$ in A level Mathematics and Further Mathematics) while some had more basic qualifications (BTEC or A level mathematics grade B or C[9]). So, for example, in addressing the topic of 'Introduction to Matrices' some students had already learned to find the inverse of 3×3 matrices and to solve systems of equations with 3 variables; other students did not yet know

[8]STACK is an open-source system for automatic computer aided assessment of mathematics and other STEM subjects; see `www.ed.ac.uk/maths/stack` for more information.

[9]In the UK, the most common qualification requirements are General Certificate of Education Advanced Level (A level) Mathematics grades A*, A, or B. Some universities admit students to engineering courses with A level grade C, or with BTEC qualifications—a BTEC is a vocational qualification studied at school or college. They tend to be work-related and are ideal for any student who prefers more practical-based learning. BTEC qualifications allow students to continue further study at university or enter the workforce.

how to add or multiply matrices. These differences extended to all topics in the course specification. In her reflections, the lecturer wrote:

> In my first year of teaching this module, I worked with students in a fairly traditional style, presenting mathematics using PowerPoint slides in lectures and helping students in tutorials to work on problems presented in problem sheets related to the topic. With 200 students, I found it difficult to address individuals or to engage with any form of discussion in lectures, or to use explicit inquiry-based tasks; although I had been able to do all of this in earlier teaching of a cohort of 50 students in Materials Engineering (the ESUM project).

In comparing the two cohorts, the size difference (200 v 50) was highly significant; the difference in student mathematical experience was significant in both cohorts, but there were more highly qualified students in the 200 cohort. It was difficult to design lecture material to suit all 200 students, and a different pedagogy was needed for experienced as for non-experienced students. She wrote further:

> I have never used the teaching approach of many of my colleagues of spending a lecture writing out the mathematics from start to finish on a large board (black or white) at the front of the lecture room. I prefer to use PowerPoint because (1) it allows me to face my students as I talk, and to actually look at them and make eye contact (as far as this is possible in a large lecture hall). Also, (2), PowerPoint allows me to animate my slides, building up mathematical formulae and relationships using the whole space of the slide, and emphasising concepts using colour, movement and timing. I talk as I animate and so there is both an oral and a visual exposition of the mathematics.

Every lecture at LU is recorded on the university system of recording all lectures for students to access as they wish. It is encouraged also to save lecture notes and slides on the course VLE (Virtual Learning Environment) page for student access. There should therefore be no need for students to spend their lecture time copying the words and symbols from the slides. Although this is often emphasised in lectures, many students ignore the message and, nevertheless, try to copy everything written. It is as if there is an unwritten rule that what lecturers write in lectures should be copied by the student for future study. The lecturer reflected:

> In teaching, I wish to engage students with the mathematics. As I talk to them, I hope they are trying to make sense of what I say, and I hope that the visual words, symbols, diagrams and animation on the slides contribute to their sense-making. I use a slow clear articulation so that students are not disadvantaged by my speaking too quickly or not finishing my words.

Feedback about this module from some students to their Engineering tutors was somewhat negative: some complained that the teaching was too slow and elementary (despite the inclusion of more challenging problems in the VLE material). Some did not like the slides, saying that there was not enough time for them to copy everything from a slide before the lecturer moved onto the next one. The lecturer commented:

> I taught this course twice in successive years. I will not do so again since I am reducing my working hours in the coming year. However, I can think about what I might do given time and support. I believe that it would be valuable to set up an innovation project as we did in ESUM to institute more inquiry-based activity - this might be possible in the TeStED programme.

An issue in following up the ESUM programme in this way would have been the lack of resources to support developmental activity. However, at about this time, the School of Science began an initiative called the 'Teaching and Student Experience Development (TeStED) Programme', which awards time and resources to teaching development. With interested colleagues, it could have been possible to apply to take part in this programme to build on the experiences in ESUM and in a further project in which student partners helped to design mathematical tasks (Catalyst— see section above). Such activity is as yet very small scale, but it is growing as the university recognises a need to promote teaching development.

These reflections above capture elements of the goals and practice of the lecturer. However, there is a tone of sadness: she has not managed to teach in a way that is more inquiry-based. We read some of the issues she faced: the size of the cohort, the very different levels of student mathematical experience and the use of a mode of delivery which students did not like. Implicit is the culture of mathematics teaching in the university: practices such as board writing are common; students are used to copying from the board for later review, they do not think of the value of reflecting on what is being presented during the presentation. In ESUM, the overt questioning approach of the lecturer had been successful to some extent in encouraging students to participate in the lecture, offering (tentative) answers to questions, and even engaging in discussion with peers when some disagreed with what had been said (Jaworski & Matthews, 2011). At the end of the reflections, the ideas for future development, following experience in ESUM, showed that despite negative experiences, she could see ways of achieving more inquiry-based goals.

As a final word here, mathematics teaching to engineering students in the university is delegated to the mathematics department, and engineering colleagues are not involved. It makes sense to us (authors of this chapter) that teaching mathematics to future engineers should acknowledge the use of mathematics in engineering. This would require collaboration between teachers in the two departments, enabling the design of tasks for students that could span the two subjects. In inquiry terms, this could involve modelling tasks in which an engineering problem is addressed through a developing mathematical model. It would, however, require serious reorganisation of teaching which, for the moment, seems unlikely. We refer readers to Chapter 8, which addresses inquiry-based mathematical modelling in a PLATINUM context.

15.7. Discussion

Our concluding section draws together all of the above, addressing how these activities, developments, research and external factors have influenced our own learning and development. In particular we will focus on how the areas of activity we described relate to the PLATINUM project.

We have indicated (above) ways in which our work has related to the three-layer model of inquiry. In the inner layer, we provided examples of tasks that were designed to involve students in inquiry. Particularly in the Catalyst project, research has shown us the important mathematics learning development experienced by the student partners who developed tasks in collaboration with mathematics education researchers. As the Foundation Studies teacher uses these tasks with her students, year by year, modifying them according to what she learns from her data, we see (in layer 2) a clear contribution to development of the Foundation Studies teaching of mathematics.

The Catalyst project embedded clear activity related to the middle layer of the model. Working with our student partners, we learned as they learned. Although the project was very small scale, we see clearly the mathematical learning outcomes of our student partners as they engaged enthusiastically with task design. Their own words are testament to the learning. We ask, how can we use this methodology with larger groups of students (50, or even 200)? We do not have an answer to this challenge, but it is something for us to work on further in our inquiry community.

In Teaching Engineering Students we see (in layer 2) a teacher overtly reflecting on her teaching and recognising ways in which her teaching practice did not, or could not, achieve what ideally she would like to be possible. One thing that this reveals is that it is hard for a teacher to try to engage alone with inquiry into teaching. Comparison with the ESUM project emphasised the value of having a research associate working alongside to gather data and stimulate reflection. The inquiry group in ESUM (four colleagues), designing, teaching and monitoring activity, was supportive both in the design of teaching (tasks and pedagogy) and in reflective inquiry which led to improvements in the course as it developed.

In the third layer we see an overt developmental intention supported by collection and analysis of data related to questions we wanted to address. The ESUM project had been one good example of this in which a CoI designed, taught and evaluated the teaching and learning in the project, with feedback to future teaching. Such activity was achieved also in the Catalyst project. Here, mathematics teachers engaged overtly in research into the practices in which they participated, addressing clear research questions. The Catalyst work is ongoing in the sense that the teacher is still building on what has been done and learned in ongoing teaching/learning development. Both projects have published articles which share learning outcomes from the inquiry activity with interested colleagues more widely.

We believe that essential to the development arising from this work is the inquiry group. When colleagues together explore (inquire into) aspects of their own teaching and learning, development takes place (both for the individual and for the community) and new knowledge emerges. When the inquiry activity is in the third layer, systematic analysis of data results in knowledge which can be shared with the wider community.

We can show the above in a diagrammatic representation of our inquiry model in PLATINUM.

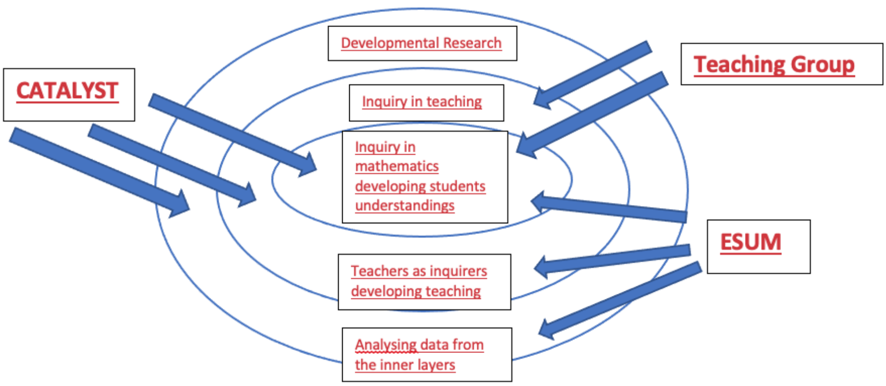


FIGURE 15.4. Linking activities to the 'Three Layers of Inquiry.'

From the model it can be seen that our inquiry activity spans all three layers. In terms of the PLATINUM intellectual outputs (IOs 1 to 6—see Chapters 2 and 5), we have focused on three of them. The inquiry model presents a theoretical perspective of the whole inquiry process (this is IO1). This encompasses our developmental activity through its three developmental stages. The central layer of the model focuses on students learning mathematics together with their teachers using inquiry-based tasks and teaching units. This is IO3. The middle layer of the model focuses on the development of teaching as we have seen in the teaching group and in both Catalyst and ESUM with the inclusion of inquiry-based tasks and teaching units, which is IO3. We see that IO3 relates to both the inner layers. It involves the creation of communities of inquiry through which colleagues work together to learn more about teaching. This is IO2. The outer layer of the model focuses on developmental research in which data is collected from a range of sources and analysed to provide results from inquiry-based practice which can be shared more widely. This relates to all three of our IOs.

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