Corresponding author: Svitlana Rogovchenko, svitlana.rogovchenko@uia.no
https://doi.org/10.5817/CZ.MUNI.M210-9983-2021-11

CHAPTER 11

Teaching Students to Think Mathematically Through Inquiry: The Norwegian Experience

Svitlana Rogovchenko, Yuriy Rogovchenko

We teach a subject not to produce little living libraries on that subject, but rather to get a student to think mathematically for himself, to consider matters as a historian does, to take part in the knowledge getting. Knowing is a process, not a product.
Jerome Bruner (1915-2016), American cognitive psychologist

11.1. Mathematics Education at the University of Agder

The University of Agder (UiA)[1] is a public university located in the southern part of Norway on two campuses, one in a larger city of Kristiansand where the university administration and most faculties are situated and another in a smaller town of Grimstad, about 45 kilometres distant from the main campus. UiA is one of the youngest universities in Norway, yet its history dates back to 1839 when the Teacher Training School was established at Holt rectory. Being one of the major driving forces for the regional development, UiA is also internationally oriented; it contributes to many international projects in education and research as a leading organisation (as in PLATINUM) or as a partner. The university is the home to about 13,000 students and 890 academic staff. It is organised in six faculties: Faculty of Engineering and Science, Faculty of Fine Arts, Faculty of Health and Sport Sciences, Faculty of Humanities and Education, Faculty of Social Sciences, School of Business and Law and has a Teacher Education Unit.

The University of Agder is acknowledged as one of the national leaders in mathematics education, mathematics teacher education, mathematics teachers continuing professional development, and mathematics education research. It has Norway's longest running master programme and the largest PhD programme in mathematics education. In the recent evaluation of education research commissioned by the Research Council of Norway,[2] The Mathematics Education Research Group at Agder (MERGA)[3] at UiA was rated as outstanding; it was granted a priority research centre status by the University of Agder in 2018. University of Agder hosts the Centre for Research Innovation and Coordination of Mathematics Teaching (MatRIC),[4] the only National Centre for Excellence in Education specialised in teaching mathematics. MatRIC is funded in 2014–2023 by NOKUT (the Norwegian Agency for Quality Assurance in Education), an independent expert body under the Ministry of Education and Research;[5] it also receives financial aid from the university.

[1] www.uia.no/en
[2] www.forskningsradet.no/en/
[3] https://bit.ly/2Y8Cntx
[4] www.matric.no
[5] www.nokut.no/en/

Mathematics is taught at UiA mainly within the Faculty of Engineering and Science as a service subject with the largest cohorts being engineering students on the campus of Grimstad, economics students and teacher candidates in Kristiansand. A handful of dedicated and hardworking mathematicians teaches a modestly sized group of bachelor students in mathematics on the campus of Kristiansand. As part of the Pure and Applied Mathematical Analysis Research group (PAMAR),[6] mathematicians also conduct research in fluid mechanics, functional analysis, ergodic theory, ordinary, partial, and stochastic differential equations, variational methods, mathematical modelling, statistics. In the report *Research in Mathematics at Norwegian Universities*[7] commissioned by the Research Council of Norway, the research in the period 2006–2010 was evaluated. With regard to the University of Agder, the homogeneity of a small mathematics group and scarce available resources were pointed out. This certainly affects the possibilities of course offer, which is not as wide as desired; for instance, there are no dedicated courses on mathematical modelling at UiA. On the other hand, due to relatively low student enrolment in several programmes, it is not economically feasible to tailor, for instance, Calculus or Linear Algebra courses to particular needs of different study programmes. For instance, Calculus courses are offered in the bachelor's programme in Mathematics, the 5-year master's Advanced Teacher Education programme in Mathematics, and the 1-year university preparatory programme; students in these three programs have different backgrounds and educational needs.

The Department of Mathematical Sciences at UiA has a long tradition of mathematics teacher education and teacher education with many students pursuing a master's or a PhD degree. Many staff in the faculty have teaching and research interests in mathematics education; they are supported in different forms by the University, Faculty, Department, MERGA, and MatRIC. A few years ago the department started a master's programme in mathematics with very small groups of 2-4 students recruited in the previous three years. A PhD programme in applied mathematics is now offered by the department but it currently has only one student, working in functional analysis; he defended his PhD thesis recently. A number of bachelor's mathematics courses for engineering students in the departments of engineering sciences and ICT on campus Grimstad are taught by a small mathematics unit composed of instructors with different backgrounds including mathematics, geophysics, astrophysics, engineering, etc. Some courses are taught to large cohorts of engineering students with different specialisation and some are tailored to special needs of specific study programs. For instance, Mathematics 1 is offered to students in five bachelor's programmes: Civil and Structural Engineering, Computer Engineering, Electronics and Electrical Engineering, Renewable Energy, and Mechatronics, whereas Discrete Mathematics is taught in the 1-year Programme in ICT, bachelor's programme in Computer Engineering, and a 5-year master's programme in Artificial Intelligence.

Traditionally, there has been very little collaboration between mathematicians and mathematics lecturers in Grimstad and Kristiansand who were separated not only by 45 km of distance between the campuses but also by their affiliation with different departments and study programs, even though within the same Faculty. The situation started improving after the Centre for Excellence, MatRIC, was established at UiA with the focus on mathematics teaching and learning for specialisations other than mathematics.

[6] `https://bit.ly/39YkuTJ`

[7] `https://bit.ly/3sTXNsv`

11.2. The MatRIC-PLATINUM Community at the University of Agder

As explained in Section 11.1, the University of Agder has a very good mathematics education environment; this contributed positively to the development of the local community of inquiry (CoI) and a larger PLATINUM community in general. Many activities organised within the PLATINUM project are especially relevant to MatRIC since the Centre focuses on mathematics teaching and learning within the university study programmes in non-mathematics disciplines such as engineering, natural sciences, economics, and teacher education. The main activities of the Centre are related to its five networks for Digital Assessment, Modelling (led by the second author), Teacher education, Simulation & Visualisation, and Video. Therefore, MatRIC supports relevant educational projects that enable sharing and development of effective use of video, digital, web-based, and emerging technologies in teaching, learning, and assessing mathematics. The Centre is very much interested in the use of most recent research discoveries in psychology and education in teaching, learning, and assessing mathematics and works to identify, understand, and evaluate effective innovation in practice.

During the first years since its establishment in the end of 2013, MatRIC arranged many interesting events including a Video Colloquium, a Mathematical Simulation and Visualisation Symposium, and a Computer Aided Assessment Colloquium. The second author organised two Mathematical Modelling Colloquia in 2015 and 2016 with invited speakers from Denmark, Germany, Mexico, the Netherlands, Norway, Portugal, Sweden, UK, and USA. These events brought together mathematics educators, scientists, engineers, computer scientists and economists in cross-disciplinary teams to produce workplace simulations and realistic tasks for mathematical modelling. Several PLATINUM team members met at these events to discuss the role of mathematical modelling in university education; these first contacts led to the alignment of research interests with the subsequent establishment of new collaborations. Not surprisingly, mathematical modelling became one of the important directions in the development of the PLATINUM project. Another important initiative taken by MatRIC was the organisation of the Mathematics Teaching Induction Course, first in collaboration with the Norwegian University of Science and Technology (NTNU) in 2015–2016 and later on in collaboration with the German Centre for Higher Mathematics Education; the most recent one was arranged in 2019–2020.[8] The experience of the first author with the organisation of the very first induction course for newly appointed and less experienced university lecturers in mathematics was very useful for the organisation of the related professional development activities in Intellectual Output 4 of the PLATINUM project (see Chapter 7).

The PLATINUM project was supported by MatRIC from the very beginning due to its relevance to the main goals of the Centre whose strategic policy envisions that effective mathematics teaching and learning result in motivated students gaining fundamental subject knowledge and understanding the important role played by mathematics in modern society. Several PLATINUM project partners met at educational events organised by MatRIC; many stimulating discussions regarding possible applications for external funding for research or educational projects were initiated there. MatRIC funded a number of partner meetings where the draft of the main ideas of the PLATINUM project were conceived and parts of the application for the EU funding through the Erasmus+ programme was prepared; this is described in more detail in Chapter 5 of this book. During the project, MatRIC and PLATINUM collaborated

[8] www.matric.no/articles/130

to provide the best educational experience to students, training them to understand better fundamental mathematical ideas and to be capable of applying these ideas for solving problems encountered in daily life and at the workplace. MatRIC's vision "Students enjoying transformed and improved learning experiences of mathematics in higher education" perfectly aligns with the goals set for the PLATINUM consortium and for the local team at UiA whose ambition is to teach students so that they enjoy mathematics and appreciate its relevance as a powerful tool for effective problem solving. Although MatRIC spans a much larger area of interests, when it comes to teaching mathematics at the university level, it is quite difficult to separate the core interests of MatRIC and PLATINUM communities due to intricate visible and invisible links between the two; therefore, we quite often refer to PLATINUM CoI at Agder as a "MatRIC-PLATINUM team."

Daily work of the PLATINUM community of inquiry at UiA has been influenced by the changes in modern views on mathematics teaching which contrast but also complement the traditional professor-centred approach. Promoting inquiry-based methodology in our teaching, we motivate students to take more responsibility for their own learning and engage actively in constructing their understanding of mathematical subjects by combining individual studies, small group work with peers, and whole class discussions. Our explorations of new ways of teaching were encouraged by the recent empirical research which reports an about 6% improvement in examination scores in active learning classes whereas students in traditional mathematics classes were 1.5 times more likely to fail the exams (Freeman et al., 2014). Remarkably, both results were consistent not only across STEM disciplines but also across different class sizes (smaller classes with fewer than 50 students perform even better).

11.3. Promoting Conceptual Understanding in a Differential Equations Course for Engineers

A lack of conceptual understanding in mathematics and a wish to skip theory in favour of framed colourful formulas in the textbook and step-by-step recommendations do-it-this-way are often characteristic in teaching mathematics to engineering students. Ditcher (2001) pointed out that quite a few engineering students take an instrumental approach to their studies with a "motivation to pass exams in order to obtain a degree (and hence a job), rather than being driven by an interest in learning" (p. 25). However, many professional engineers highly value advanced mathematical thinking. For instance, Devlin (2001) stressed that "the main benefit they [software engineers] got from the mathematics they learned in academia was the experience of rigorous reasoning with purely abstract objects and structures. Moreover, mathematics was the only subject that gave them that experience" (p. 22).

Therefore, teaching future engineers is always a challenging task that requires a compromise between theory and rigour on the one hand and procedures and applications on the other hand. For many years, university courses in Ordinary Differential Equations (ODEs) have been an important part of engineering education (Francis, 1972). The research indicates that an inquiry-oriented approach to teaching ODEs contributes significantly to students' knowledge retention (Kwon et al., 2005; Rasmussen & Kwon, 2007). Nevertheless, students' experience in difficulties distinguishing between the meanings assigned to different types of solutions (general, particular, stationary, etc.) which becomes a challenge for students' learning ODEs (cf., Arslan, 2010; Raychaudhuri, 2007, 2013).

The research suggests that "if more time were spent in classrooms with students engaged in working on cognitively demanding non-routine tasks, as opposed to exercises in which a known procedure is practised, students' opportunities for thinking and learning would likely be enhanced" (Simon & Tzur, 2004, p. 92). In this first case study, we discuss how a deeper analysis of non-standard problems on the Existence and Uniqueness Theorems (EUTs) helps students to make sense of differential equations and relate the concepts of particular and general solutions. This teaching experiment was inspired by an interesting paper by Klymchuk (2015) on the use of 'provocative' mathematics problems and by the work on students' conceptual understanding of key issues in differential equations by Raychaudhuri (2007, 2013).

Earlier research has shown that students usually form a habit of applying formulas or rules without checking conditions required for the application of procedures and theoretical results, tacitly assuming that they are satisfied. Furthermore, assessment questions are often formulated so that these conditions are automatically met, and, in most cases, students are not asked to verify them. However, "ignoring conditions and constraints might lead to significant and costly errors" (Klymchuk, 2015, p. 63). On the other hand, turning the exploration of theoretical results into inquiry can be very useful for deepening students' conceptual understanding:

> How often do we ask students to prove something only to realise that they do not yet understand the statement, let alone believe it is true? Whether you are teaching students how to develop formal proof techniques, teaching a course where proof is a routine part of the homework, or just expecting students to justify assertions informally, an inquiry-friendly option is to ask students to try examples and begin to make conjectures before writing proofs. Working through examples ensures that students understand the key definitions they will need in the proof. (Dorée, 2017, p. 181)

Although proof writing was not the goal in the course, turning standard testing of easily verifiable assumptions into challenging inquiry questions about EUTs that promote advanced mathematical thinking sounded very attractive to the authors.

Challenging the status quo, the first author, a mathematics lecturer, designed the set of six non-standard problems on EUTs aimed at enhancing the conceptual understanding of a group of 23 fourth year students in mechatronics enrolled in an Ordinary Differential Equations (ODEs) course. The lecturer's intention was to provide her engineering students with unusual situations "for which students had no algorithm, well-rehearsed procedure or previously demonstrated process to follow" (Breen et al., 2013, p. 2318). Contrary to traditional practices in mathematics courses for engineering students, problems were formulated in such a way as to engage students more deeply with important details of theoretical results focusing on the development of conceptual understanding rather than procedural skills. The lecturer wanted to explore how non-standard questions can be used to challenge students, develop their analytical skills, and contribute to conceptual understanding of important notions and ideas in an ODE course for engineering students. Furthermore, introducing the small group work in the project, the lecturer wanted to understand to what extent have individual work and group discussions contributed to students' conceptual understanding of EUTs and influenced their individual solutions submitted for assessment. The authors started to select tasks by looking up relevant material in the textbook. But this did not suffice, and they browsed related research literature for more inspirational ideas. Last but not least, the authors contacted Dr. Treffert-Thomas from Loughborough University requesting some methodological advice on the organisation of the teaching experiment. The combined efforts of two mathematicians and a mathematics educator led to the design of the final set of six problems. Having in mind both improved students learning and subsequent educational research, this small community of inquiry adopted a formative approach to research known in the literature as design-based research (Swan, 2020) where the set of the tasks has been designed, developed, and refined through several consecutive cycles of observation, analysis, and redesign, including the use of the feedback from students.

Students started by working on the problems individually, first during the tutorial time and then at home producing their own solutions to the problems (see sample problems in Figure 11.1). All problems required conceptual understanding of the EUTs and their correct application in situations that were different from those traditionally requested by most texts, where it was necessary to directly verify the assumptions and conclude whether a theorem could be applied or not. For instance, for solving the problems shown in Figure 11.1, students had to apply the theorem that states "if coefficients of a linear DE are continuous on a given interval, there exists a unique solution of the initial value problem on this interval." Students learned earlier in the course how to verify that a given function is a particular solution to a given ODE but Problems 1(a) and 2(a) (see Figure 11.1) both require to check for the general solution. This is a rather unusual problem for engineering students, not found in most standard textbooks for engineering and science students. In fact, it is not hard to verify that the given function is a solution to the given ODE (and students were able to do this) but to show that it is the general solution, one has to explain the role of the arbitrary constant (we refer to the 'first method' later on). Alternatively, one can derive the general solution using an integrating factor or variation of constants; this establishes the formula for the general solution (the 'second method').

Sample problem 1

a) Verify that $y(x) = \dfrac{2}{x} + \dfrac{C_1}{x^2}$ is the general solution of a differential equation

$$x^2\, y' + 2xy = 0$$

b) Show that both initial equations $y(1) = 1$ and $y(-1) = -3$ result in an identical particular solutions. Does this fact violate the Existence and Uniqueness Theorem? Explain your answer.

Sample problem 2

a) Verify that $y(x) = C_1 + C_2\, x^2$ is the general solution of a differential equation

$$x\, y'' - y' = 0$$

b) Explain why there exists no particular solution of the above equation satisfying initial conditions $y(0) = 0$; $y'(0) = 1$.

c) Suggest different initial conditions for this differential equation so that there will exist exactly one particular solution of a new initial value problem. Motivate your choice.

FIGURE 11.1. UiA examples of nonstandard ODE tasks.

The formulation of Problem 1(b) is also unusual for engineering students. The 'trap' was set for those who might erroneously believe that the integral curve associated with the solution $y = 2/x - 1/x^2$ passes through the two different points given as initial conditions (ICs). However, since both coefficients $p(x) = 2/x$ and $q(x) = 1/x^2$ are not defined at $x = 0$ and are continuous either on $(-\infty, 0)$ or on $(0, +\infty)$, but not on any interval including zero, two different solutions defined by the same expression exist on two disjoint intervals, each containing one of the initial points. In Problem 2(b) it was necessary to verify that both ICs cannot be satisfied because the slope of the solution to the given ODE passing through the origin cannot be equal to 1 at $x = 0$ whereas for solving Problem 2(c) one had to notice that the ICs were given at the point $x = 0$ where the coefficients of the DE have a discontinuity. Therefore, even though a solution may still exist and be unique, this cannot be deduced from the EUT since its conditions are not satisfied. It is possible to resolve this issue by modifying the ICs, namely, either by changing the initial point from $x = 0$ to any other value and use the EUT, or by modifying the ICs at zero and showing by direct inspection that the solution exists (the latter also requires the proof that the solution is unique).

After working on solutions individually, students met in small groups to discuss their individual solutions and agree on a common set of solutions to the assignment to be presented to the class. After the presentation of solutions to the entire class (each group presented their solution to one of the six problems), students were given an opportunity to work at home on the assignment finalising their individual solutions which were then submitted to the lecturer who graded the assignment and provided the feedback to the students. An important feature of this teaching experiment was the lecturer deliberately not interfering in the students' small group discussions which were organised outside the course hours; she also did not contribute to the classroom discussion when group solutions were presented, encouraging students to engage critically in the peer discussion.

The analyses of three sets of students' individual written solutions (solutions produced during the tutorial session, at home and final solutions submitted for grading) and recorded discussions in five small groups along with the audio recordings of students' final presentation of solutions and the lecturer's reflections on the activity provide a useful insight into the process of students' learning. For example, the lecturer noticed, quite unexpectedly, that students experienced certain difficulties with the correct mathematical meaning of particular and general solutions. This problem has been also reported in several research papers on students' conceptual understanding of ODEs (Arslan, 2010; Raychaudhuri, 2007, 2013). However, the students in our teaching experiment worked out collectively what the "violation of conditions of EUTs" means. They developed new understandings in this context that the lecturer did not foresee while designing the coursework. Furthermore, on some occasions, discussions within the group led students to adopt familiar routines at the expense of other ideas that could have been more appropriate and could have led to conceptual understandings. We provide two excerpts from the transcripts of self-recorded small group discussions to illustrate the success and difficulties experienced by the students. In what follows, the students are identified by two digits, so, for instance, S23 means the third student in the second small group.

Excerpt 1

S12: Since we got the solution, I just took the derivative of that and put it into the original equation, to see that two equals two, and that was the case, that was my verification.

S12: Mine as well.

S12: Mine too.

S12: So, I was the only one who actually did any work, [laughter] so I actually integrated the whole thing, and ended up with the right expression, so [. . .] your way of doing it is a lot easier.

S12: A bit more efficient at least.

S12: And I had a problem with the term in front of C_1, which should be minus, according to the task, I only got it positive because of the integration.

In this episode, four students in Group 1 discuss the solution to Problem 1(a). We notice that explaining the solution to the task, student S12 describes the verification procedure known for particular solutions and somehow 'melts together' the concepts of the general solution and particular solution using a much more general notion 'solution' and not paying attention to the important loss of meaning.

Excerpt 2

S21: How can we verify that this is the general solution?
S22: Obviously, differentiate the solution, put it into the differential equation and see if it is correct as usual.
S23: You can also say that it is a derivative, you can use the product rule to bring it together, to integrate.
S25: I did the same as you did, using the integrating factor, multiplying and then I just solved the equation because it is solvable.
S23: You did not use u times v derivative and you get it v derivative times u plus u derivative times v?
S25: Yes, I used the method for it, where you define $\mu(t)$ as the integrating factor and then multiply in, the same as we did in the first lesson.
S24: I also solved the equation by the integrating factor but I think it is easier just to differentiate it once and put it into the original equation and see if it is a correct solution.
S25: But there could be more solutions, they are not general solutions.

In the second episode, five students in Group 2 also discuss their solutions to Problem 1(a). S22 suggests the procedure to verify that a given function is a solution to a differential equation but does not explain why it is a general solution. Similarly to what was observed for Group 1, S22 also does not distinguish between the two different types of solutions. S23 concentrates his attention on particular details of the solution procedure. S24 tends to agree with S21 and the obvious lack of attention to the detail at this stage potentially leads to an incomplete solution. Reacting to this unfortunate situation, S25 tries to bring attention to other possibilities but the group mates do not recognise the importance of this suggestion and proceeded further to the discussion of the next task.

Summarising the discussion of Problem 1(a) in two groups, we observe that after the encounter with a multifaceted definition of solution in the university ODEs course (general and particular solutions, solutions to initial value and boundary value problems), different from students' previous experience in other courses, students changed their mathematical discourse and embraced new meanings of the familiar term 'solution.' Surprisingly, for many students the work with the EUTs was less confusing than the work with the fundamental for ODEs question regarding the difference between general and particular solutions. After the lecturer analysed students' written individual solutions, transcripts of small group work and presentations of solutions in the class, it turned out that students in the course can be divided into three main types with respect to the development of their skills and conceptual understanding: pseudo-learners, potential learners, and learners. This classification has been suggested by Raychaudhuri (2013).

***The learner*:** This student is in possession of a coherent cognitive structure, and tries to maintain and rebuild it on a continual basis. A student such as this acknowledges a conflict, and attempts to reorganize his or her cognitive structure while keeping all the previous connections intact. He or she may or may not be successful in this attempt, but it is his or her approach that indicates the individual's status as a learner.

***The potential learner*:** This student is in possession of a more or less coherent cognitive structure, but does not try to maintain or rebuild it on a continual basis. The student acknowledges a conflict, but does not want to go to great length to remedy it. Faced with a conflict the student often deals with it by letting go of one or more previous connections. In other words, they suppress the conflict by *patching* it with a temporary quick-fix solution

***The pseudo-learner*:** The pseudo-learner: This student stockpiles items of knowledge one after another in an almost *linear* structure where connections are primarily local (often via processes studied in a localized context). He or she will not recognize conflict (without a connected structure, questions of conflict do not arise) and will compartmentalize the conflicting pieces if they are pointed out. Either way, the conflict will cause no perturbation to their cognitive structure. (p. 1241)

We explain this rather general classification in the following table providing more specific details relevant for our example on the understanding of EUTs. The interested reader would very likely find relevant applications of this classifications to own students.

Student type	*Challenge (evidence: homework)*	*Skills development (evidence: group work & presentation)*	*Understanding (evidence: final homework)*
Student A, pseudo-learner	Did not understand the logic of EUTs.	Performed several procedural steps correctly without developing conceptual understanding.	Did not understand the difference between necessary and sufficient conditions; did not understand the essence of EUTs; submitted many incorrect solutions.
Student B, potential learner	Understood the main ideas of EUTs.	Provided mostly correct solutions without elaborating the details and without reference to theoretical results.	The final homework has been very little influenced by the discussions and presentations and contained some incomplete or inaccurate solutions.
Student C, learner	Understood the logic of EUTs but missed some important details.	Refined solutions supporting them with references to appropriate theoretical material.	Used the results of the discussions for improving individual solutions significantly.

TABLE 11.1. Classification of students on the basis of written work and oral contributions.

Looking for students' feedback on this teaching experiment, the lecturer distributed two questionnaires, in the beginning and in the end of activity. Prior to the experiment, students rated themselves as quite competent in mathematics (3.3 out of 5 on the Likert scale, 5 being the highest score, here indicating 'very competent'); they also believed they possessed mathematical knowledge sufficient for their needs as engineering students (3.8 out of 5). Reflecting about the activity, students found the tasks in the assessment interesting (4.1 out of 5 on the Likert scale), enjoyable (4.0 out of 5), and very challenging (4.4 out of 5). Most students recalled that it was nice to have discussions, both in small groups and in the class, and to be able to see and discuss alternative solutions suggested by the peers (12 out of 19). It seems that inquiry in small groups through discussions was one of the most enjoyable and appreciated components of the activity, as acknowledged in students' answers quoted below.

> Nice to have a discussion and hear other people's opinions and thoughts.
>
> The discussion was surprisingly interesting because you learn a lot when you have to explain your reasoning.
>
> I learned a lot by solving it for myself and then got alternative inputs and different ways of solving/evaluating.
>
> I found the discussion part interesting, and it was nice to see that the majority of tasks was solved in a similar way.
>
> To make individual solutions to a common problem may ultimately give a better solution in the end than to work as a group from the start.

For a more detailed analysis of the use of non-standard problems in an ODE course for engineering students, we refer the interested reader to the papers of Treffert-Thomas et al. (2018) and Rogovchenko et al. (2020).

11.4. Innovation Versus Students' Inertia and Institutional Constraints

The second episode describes a not-so-successful teaching experiment with the first-year bachelor's students in a standard Multivariable Calculus course. The course is offered to students in the Bachelor's Programme in Mathematics, Advanced Teacher Education level 8-13, the 5-year Master's Programme in Mathematics Education, and the 1-year Bridging Programme in Mathematics. The student population was quite diverse, although for most students it was their very first year at the university, there were also a few more mature students; several students had received (at least partly) school education abroad. This experiment has been conceived by the authors in collaboration with Professor Simon Goodchild, a mathematics educator, specifically with the PLATINUM project in mind. Therefore, upon our request, permission to teach the course in English was granted by the Head of the Department. During the preparation to teaching in this course, the second author carefully explored available teaching resources, searched for textbooks, both in print and online, as well as for relevant lecture notes featuring the combinations of keywords *inquiry*, *active*, and *calculus*. Unfortunately, only a few online resources were available, the most appropriate being "Active Calculus – Multivariable" prepared by Steve Schlicker and his colleagues at Grand Valley State University.[9]

The discouraging results of the literature search clearly indicated that setting a Multivariable Calculus course within an inquiry-based teaching framework would not be an easy task neither for the lecturer nor for the students. The three-fold team was meeting regularly (once or twice a week) before the course start and also during the teaching to discuss the learning goals, teaching materials organisation of lecturing, tutorials, and exams, as well as the problems for the use in the class. The mathematics educator attended most lectures; he was observing the teaching and taking notes; he also had several conversations with students regarding the course; lectures were recorded to allow for subsequent analysis. His written comments were discussed with the project team after the classes and possible adjustments to teaching were suggested to the course lecturer.

---

[9] `http://bit.ly/3bUFk9k`

In the very beginning of the course, the lecturer described to the class the goals and the organisation of the teaching and learning process. He emphasised that the course is demanding and clearly accentuated students' attention to their role as *knowledge explorers and gainers* and his role as a *team member* assisting students' learning rather than a lecturer. The lecturer thoroughly explained the peculiarities of the current course organisation. All important details regarding the course and exam organisation were discussed by the three-fold course team and carefully described in the course description posted on Canvas, the learning management system currently used at UiA.

> Learning outcomes are set at the beginning of each week. They state the knowledge and skills that the students should acquire every week and are important for students' progress through the course. [...] What is new and special about the course this semester: to facilitate students' conceptual understanding of the material and to contribute to its better retention, a form of active learning known as inquiry-based learning will be employed. This means that in addition to traditional lecturing, students will be also more actively engaged in learning during the lectures through discussions in small groups, questioning and exploration. Elements of inquiry-based learning will be also incorporated in some problems included in four non-compulsory problem sets (the total of twenty problems). Sixteen out of twenty problems will be quite similar to those in the main textbook but will be selected from the sources different from it and thus no answers or solutions to the problems will be known; four of them (one for each set) will be selected for the final written exam. Four problems out of twenty will have a distinct flavour of inquiry; one of these will be selected for the final written exam. Answers or solutions to the problems in these four sets will not be provided but students who seriously engage in their solution will receive a comprehensive feedback. The course team composed of a lecturer, an experienced mathematics education professor and an experienced mathematics professor will regularly monitor and timely adjust, if necessary, the course teaching and learning strategy and selection of teaching and learning materials.

In the first lecture, students were introduced to the SOCRATIVE app for mobile phones[10] and informed about its use during the lectures for getting fast feedback on students' progress in the course. To test the app, students were asked to answer two questionnaires, each with three questions, distributed during the break and right after the first class (see Table 11.2).

The total of 43 answers to questions 1-3 and 38 answers to questions 4-6 were received by the SOCRATIVE app; students' choices are reflected in Figure 11.2. The survey results were very encouraging and clearly indicated students' preparedness to work hard and engage. In fact, 90% of the students expected the course to be more difficult or much more difficult than other courses; 78% expected to spend at least 16 hours per week on this course; 90% claimed that attending lectures was necessary and very necessary; 86% thought that attending seminars was necessary or very necessary; 92% assumed that working on non-obligatory tasks was necessary or very necessary; and 68% expected the course to be at least moderately interesting.

Students in the course seemed to agree with the need to work harder and be engaged in so-called active learning defined by Bonwell and Eison (1991) as "anything that involves students in doing things and thinking about the things they are doing" (p. 2). Emphasising the importance of active engagement of students in learning, the lecturer also warned about specific obstacles associated with the use of active learning methodology. These would, in particular, include (1) the difficulty to adequately cover the course content; (2) limited class time available; (3) possible increase in the

[10] `www.socrative.com/`

amount of preparation time; (4) the difficulty of using active learning in large classes; and (5) a lack of materials, equipment, or resources (Bonwell & Eison, 1991). In fact, the lecturer of the course and the two professors supporting him experienced all these factors, acknowledging that the organisation of active learning in a medium-size class represents a serious challenge.

Nevertheless, the team worked enthusiastically in the hope that the positive students' feedback to the survey will be also supported by their increased effort in learning the material in the course. To stimulate students' engagement with the material, the lecturer was suggesting quizzes with 1–3 problems for "discussion with a peer sitting

Question	*Possible answer*
Q1. When you compare this course with other courses you take; do you expect this course to be:	A Much more difficult B More difficult C About the same level of difficulty D Easier E Much easier
Q2. To be successful in this course, an average student is expected to work on course tasks outside of classes for 16–20 hours each week. How many hours do you expect to spend, studying this course outside classes, to be successful?	A. More than 25 hours each week B. About 20 hours each week C. About 16 hours each week D. About 12 hours each week E. Less than 7 hours each week
Q3. In your opinion, how necessary is it to attend the lectures to ensure success?	A. Very necessary B. Necessary C. No strong feeling D. Not necessary E. A poor use of my time
Q4. In your opinion, how necessary is it to attend the seminars to ensure success?	A. Very necessary B. Necessary C. No strong feeling D. Not necessary E. A poor use of my time
Q5. In your opinion, how necessary is it to work on all the tasks and problems, which are not obligatory, to ensure success?	A. Very necessary B. Necessary C. No strong feeling D. Not necessary E. A poor use of my time
Q6. How interesting do you expect the course to be?	A. Very interesting B. Moderately interesting C. No feeling either way D. Rather uninteresting E. Very uninteresting

TABLE 11.2. Questions and possible answers in two surveys.

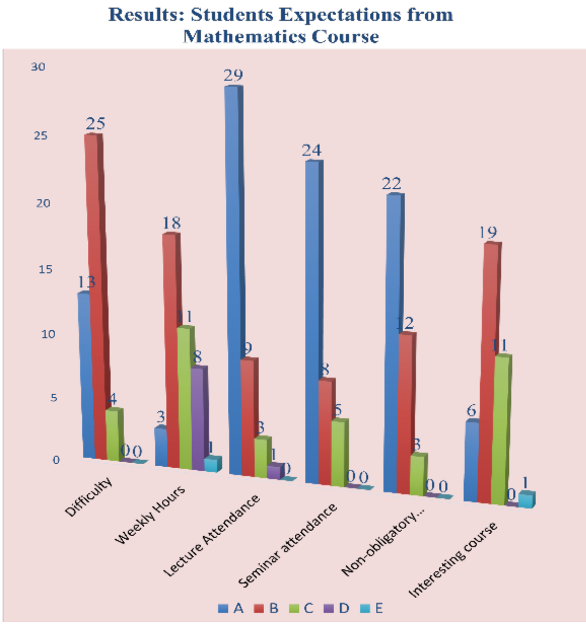

FIGURE 11.2. Students' answers to the six questions in Table 11.2.

next to you" two-three times during the lecture. The tasks required conceptual understanding of the material and very little or no computation. Students had to submit individual answers after 5–7 minutes of discussion with a classmate. The progress with the answering the tasks was projected on the screen and correct answers were marked with green bars. The student names were not visible to the class, only to the lecturer, who usually praised at the end students who answered questions correctly. The lecturer also commented shortly on the answers providing a short argument leading to the correct answer. Examples of the tasks are provided in Figures 11.3 and 11.4

(a) $$\int_0^{\pi/4} \sqrt{1 - \sec^4 x}\, \mathrm{d}x$$

(b) $$\int_0^{\pi/4} \sqrt{1 + \sec^4 x}\, \mathrm{d}x$$

(c) $$\int_0^{1} \sqrt{\frac{\pi}{4} + \sec^4 x}\, \mathrm{d}x$$

(d) $$\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, \mathrm{d}x$$

(e) $$\int_0^{\pi/4} \sqrt{1 + \sec^2 x \tan^2 x}\, \mathrm{d}x$$

FIGURE 11.3. Which integral gives the arc length of the curve $y = \tan(x)$ between $x = 0$ and $x = \pi/4$.

A Multivariable Calculus course at the University of Agder, like similar courses across the globe, is traditionally shifted towards computational aspects; this is often

$$\text{(A)} \quad \begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases} \quad 0 \le t \le 2\pi \qquad \text{(B)} \quad \begin{cases} x = 3\cos t \\ y = -2\sin t \end{cases} \quad 0 \le t \le 2\pi \qquad \text{(C)} \quad \begin{cases} x = 2\sin t \\ y = -3\cos t \end{cases} \quad 0 \le t \le 2\pi$$

$$\text{(D)} \quad \begin{cases} x = -2\cos t \\ y = 3\sin t \end{cases} \quad 0 \le t \le 2\pi \qquad \text{(E)} \quad \begin{cases} x = 3\sin t \\ y = 3\cos t \end{cases} \quad 0 \le t \le \pi \qquad \text{(F)} \quad \begin{cases} x = 3\sin 2t \\ y = 3\cos 2t \end{cases} \quad 0 \le t \le \pi$$

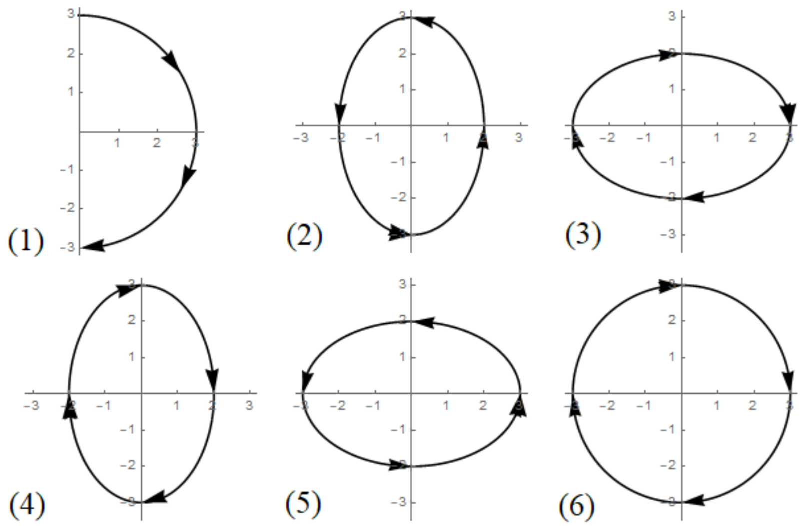

FIGURE 11.4. Which parametric equations A-F describe parametric curves plotted in Figures 1-6?

emphasised in most textbooks and in the teaching based on these texts. Not surprisingly, many students tend to memorise the formulas and algorithms without making an effort to understand them; problem solving in the class and at home frequently turns into predefined routines "repeat the steps after the lecturer" or "follow the procedure in the textbook's example." The empirical research indicates that even a simple reformulation of a traditional task as a question is useful for initiating students' inquiry and stimulating their learning.

> One step in teaching students to ask questions is to rephrase routine textbook exercises as questions that can be worked on in groups during class. This remarkably simple-to-implement shift can transform routine procedural exercises into questions that spark students' interest, deepen their conceptual understanding, encourage students to connect multiple perspectives, and inspire students to ask their own question (Dorée, 2017, p. 180).

The second author designed both tasks with the purpose of attracting students' attention to the key details important for the conceptual understanding of the material, taking inspiration from a limited selection of inquiry-oriented tasks available on the web. In the first, a slightly easier problem, students were asked to compare several possible answers which were intentionally designed to be alike; the choice of the correct one, the answer (b), requires the analysis of the main components in the equation for computing the arc length of a curve defined in Cartesian coordinates; to this end one needs to recall the general formula for the arc length along with the derivative of the tangent function. Although the assignment is not particularly difficult, 13 students out of 18 who registered for the class on Socrative submitted the answers and only 6 (about 46%) turned out to be correct.

The second task is much more challenging and requires students to associate six equations of parametric curves with their graphs. The students in their first year of bachelor's programmes did not see many similar examples, if any at all. In this task students have to pay attention to the intervals where the parameter t is defined in order to correctly identify the initial point and the direction of the motion along the parametric curve. If one does not check the details carefully, it is quite easy to confuse similarly looking graphs 1 and 6, 2 and 4, and 3 and 5. Since all parametric equations are also akin, this adds even more confusion to the task. Not surprisingly, only 4 out of 34 students (less than 12%) correctly paired all six equations with their graphs, six students made one mistake whereas quite a few students either did not attempt the solution at all, or did so only for the first few pairs.

Despite the lecturer's enthusiasm and willingness to engage students actively in learning mathematics supported by the generous advice from his two colleagues, both with extensive teaching and research experience in mathematics and mathematics education, the experiment, unfortunately, did not last long. Students' apparent understanding of the peculiarities of the course and the necessity to actively engage in learning did not help to change their reluctance to experience something new and challenging. Soon after the first few classes, a group of students complained to the study adviser and the department's head about the lecturer's too high expectations with respect to students' previous knowledge, their performance in the course, and a fear of receiving lower grades in Calculus II in comparison with top grades in Calculus I. Students also shared their concerns with the lecturer focusing, however, mostly on the language issue rather than on the lecturer's excessive demands regarding previous mathematics knowledge. Even though the lecturer reassured students that everything should settle down soon and they will receive all support needed to master the material, students were not convinced; the initiative of the mathematics educator to mediate the rising tension in a meeting arranged separately with students did not help. By the end of the second week of teaching, the head of the department—after several rounds of discussions with the lecturer, the mathematics educator involved in the experiment, the student adviser, and the study program leader—yielded to students' pressure and decided for the teaching to return to a traditional form, and we regretfully confirm that the experiment failed.

11.5. Lessons Learned

One of the distinctive features of both examples of teaching practice discussed in Sections 11.3 and 11.4 is that the authors were keenly interested not only in providing students with the learning opportunities to facilitate and promote conceptual understanding of mathematics but also in their own professional development as mathematics teachers as well as in contributing to mathematics education research. This is why, in both episodes described, the authors carefully looked up and analysed relevant research literature and asked active education researchers for methodological support. As fairly noticed by Jaworski (2006), "theory cannot show us what teaching should involve, but teachers and educators can search for clearer understandings of what teaching might involve; thus, we learn about teaching with the possibility to develop teaching" (p. 189). In both teaching experiments inquiry was used as a developmental tool and the authors worked with the mathematics educators in small communities of inquiry as described by Jaworski (2006) although with rather different arrangements. In the first case, the team was relying on the methodology of design research (Cobb, 2000) where cycles of design, testing, analysis, and redesign of the tasks over several academic terms were planned with the ultimate goal of creating knowledge for practitioners and mathematics education researchers. During the teaching experiment reported in Section 11.3, three team members met on a few occasions to discuss the design of the tasks and experiment settings and more frequently later on for the analysis of the learning activity and its redesign (the latter is not discussed in the chapter). In the second example (Section 11.4), the project team was prepared to work intensively during the entire academic term with regular meetings, extensive preparatory and follow up work, and a very active engagement of the mathematics education professor. This teaching experiment was designed primarily with the PLATINUM goals in mind and further plans for redesign and possible replication in partners' CoI. Both case studies described in this chapter fit the inquiry model in three layers (see Chapter 2). In the central layer, we have students engaging in inquiry in differential equations individually and with their peers, and in inquiry in calculus with their peers and the lecturer. In the middle layer, both authors engage in professional inquiry aimed at creating new learning opportunities for students. Finally, in the outer layer, the authors inquire with mathematics educators in wider communities of inquiry discussing implications of teaching experiments and creating new knowledge for professional use and professional development of university mathematics lecturers.

Did the outcomes of the two teaching experiments with different groups of students in different departments surprise us? The honest answer is: "not much," we knew well about possible gains and risks before we planned teaching experiments. The maturity of the group of engineering students in a graduate course and students' enhanced motivation contributed positively to the success of the first teaching experiment reported in Section 11.3; most students appreciated new learning opportunities created for them by the first author. On the other hand, in the second teaching experiment, after only five months at the university, many first-year students were not well enough prepared to unusual educational explorations; the fear of not being successful in the course with innovative elements turned out to be stronger than the wish to try new possibilities for learning differently through a more challenging and active engagement. Quite rapidly this fear developed into a panic for some students; they started seeking protection from innovation with the people responsible for the study program in the department which eventually led to the termination of the experiment.

In a very recent survey, Børte et al. (2020) recognised that "Higher Education institutions are, however, not always organised, structured, and led in ways that support and facilitate new approaches to teaching" (p. 11). They identified the existing barriers to active learning grouping them under three themes: (1) Leadership and organisation, (2) Teaching competence and training needs, and (3) Technology (ibid, p. 11). In our case, the most important factor which negatively affected the teaching experiment in Section 11.4 was related to the first theme: Although the team consisting of three professors carefully planned the experiment and the lecturer had sufficient experience with teaching Calculus courses using the same textbook for many years in Cyprus and Sweden, the department yielded to students' demands and requested to terminate the teaching experiment already in the end of the second week of teaching. Furthermore, analysing the prerequisites for student active learning to succeed as reported in the research literature, Børte et al. (2020, p. 11) identified the following three key components: (1) better alignment between research and teaching practices,

(2) a supporting infrastructure, and (3) staff professional development and learning designs. It seems that all three key components were in place in both teaching experiments, yet the first-year bachelor's students were much more reluctant to engage with active learning in Multivariable Calculus than the fourth year seniors in a Differential Equations course.

Summarising the lessons learned in the two cases discussed in this chapter, we confirm without hesitation that "the reform of instructional practice in higher education must begin with faculty members' efforts. An excellent first step is to select strategies promoting active learning that one can feel comfortable with" (Bonwell & Eison, 1991, p. vi). However, the very different outcomes in the two cases suggest that the wish, however strong, of the faculty to reform the classroom practice by introducing elements of inquiry-based learning is only a necessary, but not a sufficient condition. The most pronounced differences in the two teaching experiments are related to students‘ motivation for studying mathematics and interest in the subject, their academic maturity and readiness to innovation, and institutional support (or the lack of such).

Acknowledgements

We thank our colleagues Professor Simon Goodchild from MatRIC, Dr. Stephanie Treffert-Thomas from the Loughborough University, UK, and Dr. Paul Hernandez Martinez from the Swinburne University of Technology, Australia, for useful discussions. The teaching experiment reported in Section 11.3 was supported by the small grant no. 170803 from the Norwegian Centre for Research, Innovation and Coordination of Mathematics Teaching (MatRIC) to the first author.

References

Arslan, S. (2010). Do students really understand what an ordinary differential equation is? *International Journal of Mathematical Education in Science and Technology*, *41*(7), 873–888. doi.org/10.1080/0020739X.2010.486448

Bonwell, C. C., & Eison, J. A. (1991). *Active learning: creating excitement in the classroom.* ASHE-ERIC Higher Education Reports. ERIC. https://files.eric.ed.gov/fulltext/ED336049.pdf

Børte, K., Nesje, K., & Lillejord, S. (2020). Barriers to student active learning in higher education, *Teaching in Higher Education.* doi.org/10.1080/13562517.2020.1839746

Breen, S., O'Shea, A., & Pfeiffer, K. (2013). The use of unfamiliar tasks in first year calculus courses to aid the transition from school to university mathematics. In B. Ubuz, C. Haser & M. Mariotti (Eds.), *Proceedings of the 8th congress of the European Society for Research in Mathematics Education* (pp. 2316–2325). ERME. http://erme.site/cerme-conferences/

Cobb, P. (2000). Conducting classroom teaching experiments in collaboration with teachers. In R. Lesh & E. Kelly (Eds.), *New methodologies in mathematics and science education* (pp. 307–334). Erlbaum.

Devlin, K. (2001). The real reason why software engineers need math. *Communications of the ACM*, *44*(10), 21–22. doi.org/10.1145/383845.383851

Ditcher, A. K. (2001). Effective teaching and learning in higher education, with particular reference to the undergraduate education of professional engineers. *International Journal of Engineering Education*, *17*(1), 24–29.

Dorée, S. I. (2017). Turning routine exercises into activities that teach inquiry: A practical guide. *PRIMUS*, *27*(2), 179–188. doi.org/10.1080/10511970.2016.1143900

Francis, D. C. (1972). Differential equations in engineering courses. *International Journal of Mathematical Education in Science and Technology*, *3*(3), 263–268. doi.org/10.1080/0020739700030307

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410–8415. doi.org/10.1073/pnas.1319030111

Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, *9*(2), 187–211. doi.org/10.1007/s10857-005-1223-z

Klymchuk, S. (2015). Provocative mathematics questions: drawing attention to a lack of attention. *Teaching Mathematics and Its Applications*, *34*(2), 63–70. doi.org/10.1093/teamat/hru022

Kwon, O. N., Rasmussen, C., & Allen, K. (2005). Students' retention of mathematical knowledge and skills in differential equations. *School Science and Mathematics*, *105*(5), 227–239. doi.org/10.1111/j.1949-8594.2005.tb18163.x

Rasmussen, C., & Kwon, O. N. (2007). An inquiry oriented approach to undergraduate mathematics. *Journal of Mathematical Behavior*, *26*(3), 189–194. doi.org/10.1016/j.jmathb.2007.10.001

Raychaudhuri, D. (2007). A layer framework to investigate student understanding and application of the existence and uniqueness theorems of differential equations. *International Journal of Mathematical Education in Science and Technology*, *38*(3), 367–381. doi.org/10.1080/00207390601002898

Raychaudhuri, D. (2013). A framework to categorize students as learners based on their cognitive practices while learning differential equations and related concepts. *International Journal of Mathematical Education in Science and Technology*, *44*(8), 1239–1256. doi.org/10.1080/0020739X.2013.770093

Rogovchenko, S., Rogovchenko, Y., & Treffert-Thomas, S. (2020). The use of nonstandard problems in an ordinary differential equations course for engineering students reveals commognitive conflicts. In S. S. Karunakaran, Z. Reed & A. Higgins (Eds.), *Proceedings of the 23rd annual conference on research in undergraduate mathematics education* (pp. 1141–1145). The Special Interest Group of the Mathematical Association of America (SIGMAA) for Research in Undergraduate Mathematics Education. http://sigmaa.maa.org/rume/RUME23.pdf

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*(2), 91–104. doi.org/10.1207/s15327833mtl0602_2

Swan, M. (2020). Design research in mathematics education. In: S. Lerman S. (Ed.), *Encyclopedia of Mathematics Education* (pp. 192–195). Springer Verlag. doi.org/10.1007/978-3-030-15789-0_180

Treffert-Thomas, S., Rogovchenko, S., & Rogovchenko, Y. (2018). The use of nonstandard problems in an ODE course for engineers. In E. Bergqvist, M. Österholm, C. Granberg & L. Schuster (Eds.), *Proceedings of the 42nd conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 283–290). IGPME. https://bit.ly/3eSjnru