Inquiry in University Mathematics Teaching and Learning: The PLATINUM Project
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CHAPTER 7

Methods and Materials for Professional Development of Lecturers

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7.1. Introduction

Traditional lecturing in universities is still a common teaching practice, although research has shown that lecturing on its own is often not sufficient and leads under the existing examination conditions to surface learning (Biggs, 2003; Freeman et al., 2014). In many science and engineering programs, mathematics is still learned mainly procedurally instead of having a purposeful balance between procedural and conceptual learning (Mason et al., 2010). Occasionally there is the belief that being able to teach mathematics is automatically acquired along with years of teaching (Chalmers & Gardiner, 2015) and the unsuccessful learning results are more or less the necessary consequence of untalented students. In the professional development programmes that are organised together for university lecturers from different disciplines, there is usually no specific focus on mathematics education at university level. Pritchard (2010) argues that lecturing in mathematics has three functions that should be considered: (1) communicating information, (2) modelling problem solving including heuristic reasoning, and (3) motivating students. Mason and Johnston-Wilder (2006) point to the great importance of students' active learning involvement in the learning of mathematics and the important role of learning tasks that initiate mathematically fruitful activities, stimulate student involvement and support the development of mathematical thinking. Inquiry could be understood as a form of collective intellectual engagement. It intends to help students to gain a deeper understanding by recognising problems, searching for answers on their own, applying different heuristics and discussing them with their peers.

For collaborative inquiry with the aim of achieving a deeper understanding of mathematics, the lecturer might use appropriate students' learning tasks called IBME (Inquiry-Based Mathematics Education) tasks. Learning what a good IBME task is and how a lecturer should design and apply it in his or her class is therefore very important. But there is still practically very little or no opportunity to learn how to do this. Reflection is an indispensable element for good education also specific in university mathematics teaching practice. Supporting lecturers in their reflection on IBME tasks is very important as the explorations and discussions possibly develop their critical thinking. Moreover, reflection and a collegiate approach can support lecturers in their professional development towards IBME.

In the following section, we will first outline how PLATINUM Professional Development is presented and framed from the project's point of view. In doing so, we refer to the three-layer model that is introduced and explained in Chapter 2. Against this background, we then describe the concept and implementation of three professionalisation workshops organised in the project. These took place in Hannover, Agder, and Madrid, and pursued partly common goals, but also specific goals adapted to local conditions. Finally, we summarise the respective experiences and conclude the chapter with a discussion of some consequences.

7.2. Professional Development in IBME

Co-learning in Communities of Inquiry effectively supports lecturers in IBME and fosters their professional development in teaching mathematics (Goodchild et al., 2013). This way teaching of mathematics at university level supports the aim to achieve students' conceptual learning of mathematics. The theoretical model of IBME in higher education by (Jaworski, 2006, 2019) introduces three levels which all approach teaching and learning through the developmental principles and the interaction (see Figure 7.1).

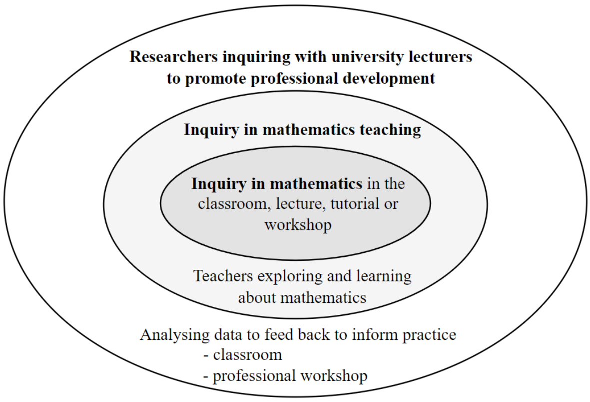

FIGURE 7.1. Three-layer model of inquiry regarding professional development.

The first layer describes the inquiry in mathematics that is carried out by students and their teacher in the classroom. Here interactions student–teacher and student–material (task) are essential. In the second layer, teachers reflect on the process in the first layer and the learning tasks are designed and adapted based on the experiences in the classroom (first layer). The lecturers discuss teaching and learning in a safe/intimate environment, give and receive feedback on the design of the learning tasks and their implementation. In the second layer, the development of the lecturers takes place in a co-learning process often together with co-creation of IBME tasks.

Next to the lecturers, in this layer more experienced members and invited experts or didacticians can promote learning and support professional development of lecturers. Indirectly these activities could improve the quality of teaching and learning in the first layer. The third layer is the evidence based layer. In this layer the didacticians and educational researchers reflect together with the lecturers on the developmental process that takes place in the second level, which performs the developmental research. From this level, they also support and intend to stimulate the reflective and evidence-informed teaching attitude of the lecturers and promote different categories of reflections on teaching and learning process. The boundaries of the first and second and second and third layers are crucial nodes in the development process because they are the communication and critical reflection / feedback nodes connecting a teacher and students, and teachers and support staff and peers, respectively. Participation in the IBME Community of Inquiry supports lecturers in their professional development.

One of the goals of the PLATINUM project is to develop and pilot a platform for the professional developments of mathematics lectures on a regular basis in the format of a "hands-on" workshop. The need for such training platform reflects the current situation in university mathematics education: mathematics lecturers often have limited or no access to the information about contemporary pedagogic and didactic methods which, in turn, contributes to the lack of motivation to use them. The research confirms that the knowledge of teaching methods, that is, a command of various teaching methods and the apprehension of when and how to apply each method has a positive impact on students' achievements (Voss et al., 2011). Looking for the possibilities of introducing a larger community of university mathematics teachers to IBME, the PLATINUM project has included in its structure the development of this topics Intellectional Output by offering methods and materials for professions development of lecturers (see the description of IO4 in Section 2.5, and PLATINUM team members piloted workshops for local communities in three countries, namely Germany, Norway, and Spain.

Three professional development workshops on Inquiry Based Mathematics Education for university lecturers of mathematics are described in this chapter. The workshops were organised at different universities in different countries. Interestingly, all three workshops focused on task development, albeit in some different ways. This is not really surprising, because tasks are the central activity drivers for students. Tasks can also be changed without fundamentally modifying the rest of a course, such as the lecture and tutorial structure, or its basic pedagogy. The change of the course organisation is hardly possible under the legal framework conditions, especially the stipulations in examination and study regulations. And the change of the basic pedagogy would require an effort that most teachers cannot or do not want to cope with due to the already heavy workload. In view of this, the (re-)designing of tasks represents a rather local change in a course. Anyway, this raises the question of how or in which directions tasks should be developed or modified, in essence the question of what makes a task an IBME task.

The workshops dealt with this question in different ways. In Hannover, dimensions and qualities of IBME tasks were introduced in advance and illustrated with examples (see, for example, Table 7.1). In Agder, on the other hand, IBME was presented as a teaching strategy together with its goals and, against this background, a reflection on suitable tasks was initiated. The Madrid workshop similarly focused on the inquiry-based task means in mathematics and the design of inquiry pathways with complex tasks. Thus, in all workshops, university mathematics lecturers were learning how to develop IBME tasks. In the final part of this chapter we analyse the workshops from the perspective of the three-layer developmental model and in view of the local contexts.

7.3. IBME Workshops

This section describes the workshops in Hannover, Agder and Madrid. In each case, we first go into the local institutional context, goals and the workshop concept based on them. Then the organisation and central contents are described by way of example. In a concluding section, the respective experiences and results are discussed.

7.3.1. IBME Workshop in Hannover, Germany.

Institutional context and goals. In Germany, as in many other countries, the content and examination requirements of mathematics teaching are largely fixed, both in the mathematics courses for majors and in service lectures for engineering courses

(Bosch et al., 2021). Introductory courses and their organisation are often characterised by high numbers of students, usually 200 to 600, and by numerous parallel exercise groups led by research assistants or tutors, each generally with more than 30 students, also because resources are limited. In addition, substantial modifications to the content, such as a reduction in breadth in favour of depth, are difficult, because subsequent courses in all degree programmes, such as advanced courses in mathematics or theory-based engineering courses for example, require knowledge of the topics dealt with. Altogether, this considerably restricts the possibilities of making changes to the methodological-didactical and content-related organisation of teaching. Given that the scope for design changes is considered to be limited, it is not surprising that general training opportunities in the didactics of higher education are hardly ever used by mathematics lecturers, and are often experienced as not very appropriate for introductory courses. Another reason is that the general lecturers' trainings do not really address mathematics teaching, which is different in many aspects, also because the nature of the content and the teaching objectives differ.

The concept of the PLATINUM workshop held in Hannover was developed with these boundary conditions in mind. Possibilities for the further development of teaching were seen particularly at the level of tasks. Modified tasks can easily replace previous tasks in regular teaching if they meet the curricular requirements. This should make it possible to address IBME aspects within the context of existing course structures, i.e., without extensive and usually hardly realisable new concepts for courses and their contents. Without neglecting the importance of pedagogical and institutional-societal constraints for the implementation of IBME, our approach was based on the assumption that "…the deliberate neglecting of topic-related aspects can lead to shortening, but above all to an underestimation of the teachers' methodological freedom to act and their subject-specific didactic decisions" (Reichel, 1995, p. 180).[1] Thus, the aim of the workshop was to provide teachers with resources to further develop existing materials, especially tasks, with regard to IBME on the basis of subject-specific analyses.

With a view to the workshop to be developed, a relevant representative of the Centre for Higher Education Didactics at the University of Hannover was invited to give a lecture in advance, in which, on the one hand, existing formats of higher education didactic training at the University Hannover were presented and, on the other hand, subject-specific possibilities for expansions were discussed. As a result, it was found that units in which tasks, their objectives and design, are discussed could be integrated into existing training programmes without any problems, whereby one could even consider that lecturers bring along their own tasks.

The workshop in Hannover was therefore aimed at providing lecturers with an offer to support them in developing tasks in the perspective of IBME or in modifying existing tasks with a view to this. The reflection of tasks, their potentials and goals was anyway a current issue in the department, since several projects were already developing digital courses to provide extra support for students in their first year of study, and these consist to a large extent of STACK tasks.[2]

[1] "…dass das bewußte Ausblenden stoffbezogener Aspekte zu *Verkürzungen* führen kann, vor allem aber zu einer *Unterschätzung* fachmethodischer Spiel- und Handlungsfreiräume der Lehrer und ihrer fachdidaktischen Entscheidungen." [Emphases as in the original]

[2] STACK is an open-source online assessment system for mathematics and STEM, which is available for Moodle, ILIAS and as an integration through LTI. It was introduced by Chris Sangwin (University of Edinburgh), see also `https://stack-assessment.org/`.

Organisation and contents. In accordance with the institutional context and mentioned goals the workshop was structured as follows:

(1) In an introduction, central ideas of IBME were first presented. These were made concrete with regard to dimensions of tasks. A tool in the form of a table (see Table7.1) was developed and made available to encourage reflections on the IBME character of a task and to guide the further process. Of course, it must always be taken into account that a task is not or is not IBME per se, but that this must always be assessed in the overall context of a course, its contents and objectives.
(2) In a subsequent second part, concrete example tasks were discussed. The starting point in each case was formed by conventional tasks from courses, which were further developed with IBME in mind. The table presented in the introduction was used to round off each of the tasks and to reflect on their IBME character.
(3) In the run-up to the workshop, the participants were asked to bring along their own tasks which they would like to discuss and, if necessary, modify. In the third part of the workshop, the tasks brought along were first presented and curricularly embedded in their respective courses. Against this background, they were then discussed with regard to further possibilities for development in the sense of IBME and, in particular, classified by means of the aforementioned table.
(4) In a final round, all participants exchanged their experiences in the context of the workshop. This naturally also involved criticism and the possibility of further development, both with regard to the organisation of the workshop and with a view to the tools provided for further development of tasks.

In the following we will go into a few details of (1) and (2). Regarding (1) we describe characteristics of inquiry-based tasks represented in the table mentioned. With regard to (2) we outline two examples that were presented at the workshop.

Characteristics of inquiry-based tasks
To identify characteristics of inquiry-based tasks we considered the following dimensions:

(a) Openness of tasks,
(b) Enabling specific inquiry strategies,
(c) Enabling discourses on techniques,
(d) Enabling inner-mathematical knowledge linking,
(e) Enabling interdisciplinary knowledge linking.

On (a): A task can be set rather open or closed. We call a task process open if it allows for different solution strategies, approaches or techniques, open-ended if it does not have a clear solution and content-open if knowledge from different areas can be profitably brought into the processing of the task.

On (b): It is addressed whether a task is (only) about applying techniques or also about conveying mathematical heuristics and developing solution strategies. Following Schoenfeld and Sloane (2016) frequently used heuristics are: draw a diagram if at all possible; examine special cases; try to simplify the problem; consider essentially equivalent problems. Regarding problem-solving cycles Mason et al. (2010) highlighted the activities: specialise, generalise, conjecture, convince.

On (c): Discourse on techniques is understood to refer to the fact that a task is not (only) about the correct application of techniques or calculations, but also about describing, discussing and questioning these techniques or calculations: What possibilities does a certain technique offer in comparison to another? When is the use of a certain technique appropriate? On (d): Inner-mathematical knowledge linking aims at connecting different mathematical concepts and overcoming the compartmentalisation of knowledge.

On (e): Interdisciplinary knowledge linking aims at connecting mathematical and extra-mathematical concepts.

We have compiled the dimensions and their aspects in a table (Table 7.1) and in this way made them available to the participants of the workshop. In view of a particular task, the table could be used to generate questions about its IBME characteristics and further development possibilities. In the next section we exemplify how we arrive at a kind of IBME profile for tasks.

Content dimension / **Task format**	***inquiry*-strategies** are to be learned	a **discourse on techniques** should be stimulated	**inner-mathematical knowledge linking** is to take place	**interdisciplinary knowledge** is to take place
Closed				
Process-open				
Open-ended				
Content-open				

Table 7.1. Table of dimensions for inquiry-based tasks.

Examples

The first example is a task developed for first-year student teachers. Its development is based on the assumption that the students know 'curve discussion' from school essentially as a computational and procedural application of criteria. Accordingly, the idea of the task is that a purely computational and procedural approach does not always lead to desired results. Instead, definitions must be consulted and applied. Although such procedures are addressed in school textbooks, they are rarely used in the classroom. Thus, the aim of this task is to problematise the fixed focus the procedural and to draw attention to definitions and concepts. The task is discussed in detail in Chapter 3 of this book.

The second example presents a somewhat more advanced task that might be given in an Analysis I course for weekly work. The basis for the development of this task was an explorative subject-specific analysis (Hochmuth, 2020) of a classical theorem by Kahane (1961). This theorem answers in the one-dimensional case for continuous functions on the interval with regard to the maximum norm the question about necessary and sufficient conditions for the convergence order $O(n^{-1})$ if one allows piecewise constant functions with at most n subdivision points, which otherwise may be chosen arbitrarily in the interval. The answer was that this exactly applies for functions of bounded variation. Obviously, Kahane's Theorem as such is too complex for an introductory course in Analysis 1. However, this does not apply to the idea and questions underlying the theorem. Accordingly, in the workshop the potential of the context was discussed in view of the objective to make basic courses more meaningful and relevant for students of mathematics. Rationales for the treatment of content in basic mathematical courses, here for example the notions of bounded variation or approximation order, are often not clear to students, but can be exemplarily explained against the background of Kahane's theorem.

With regard to functions of bounded variation typical tasks in Analysis 1 are limited to questions of the following type (Heuser, 2013): Show, that the function $g(x) = x\cos(\pi/x)$ when $x \neq 0$ and $g(0) = 0$ is continuous on $[0, 1]$, but not of bounded variation. Or: Show that the variational norm possesses the norm properties. These tasks are in no way 'bad.' They address basic techniques that students should have at one's disposal. However, the tasks are limited in terms of understanding the concepts and their meaning. So the point made here is that all tasks in Analysis 1 about bounded variation are more or less of this type.

Based on Kahane's Theorem and with a view to the intended openness of the task format various tasks can be formulated. A rather open version is represented by the following:

> Consider continuous functions on the interval $[0,1]$. To approximate such functions, you can use piecewise constant functions on $[0,1]$ with respect to any subdivisions. Consider the error regarding the maximum norm. How is the convergence order related to the smoothness of the function? Characterise differences or extensions to the approximation ideas known to you so far. Find contexts in which such ideas play a role (inner- and/or extra-mathematical).

The IBME profile of this task is represented in Table 7.2.

Content dimension / Task format	***inquiry*-strategies** are to be learned	a **discourse on techniques** should be stimulated	**inner-mathematical knowledge linking** is to take place	**interdisciplinary knowledge** is to take place
Closed				
Process-open		✓		✓
Open-ended	✓	✓	✓	✓
Content-open	✓	✓	✓	✓

TABLE 7.2. Table of dimensions for the IBME task related to Kahane's Theorem.

The appropriate formulation of a task is, of course, dependent on the students' level of knowledge. In (Barquero et al., 2016) problems with tasks that are to some extent too open are discussed. A task which is much more closed but still has IBME character is for example as follows:

> Given $f \in C[0,1]$ with $f(x) = x^\alpha$ for $0 < \alpha < 1$. How well can these functions be approximated in terms of piecewise constant functions and uniform subdivisions? What is a good choice for the respective constants in this case? Why? Is there a better choice of subdivision points? Is there a fixed sequence of subdivision points with the convergence order $O(n^{-1})$? Is there a sequence of subdivisions with even faster convergence? If necessary, how must the choice of subdivision points be adapted? The respective choice of subdivision points assumes that you know $0 < \alpha < 1$. Is it possible to find a procedure that provides $O(n^{-1})$ for all $0 < \alpha < 1$? Characterise differences or extensions to the approximation ideas known to you so far. Find contexts in which such ideas play a role (inner- and/or extra-mathematical).

After the presentation of the table and the explanation of its use with examples, the participants of the workshop presented the tasks they had brought with them. These were first discussed together with regard to their IBME character and then against the background of the respective teaching-learning situation with regard to possible changes or extensions. In these lively discussions, it became apparent that the tasks brought with them, in terms of their content, already aimed at conceptual learning. However, it also turned out that the potentials of the tasks for conceptual learning could be more precisely specified in the course of the discussion about their IBME character and that various proposals for their expansion could be worked out.

In the final round, the participants rated the workshop as overall positive, especially the joint discussion based on the tasks they had brought with them. The common work on these enabled the non-didacticians to see a value in the terminology and the table. However, it also became clear that it would definitely be a problem in everyday life to think about tasks in such detail and for such a long time. Another important finding was that it is not a problem for non-didacticians to discuss and reflect on tasks from a subject-specific-didactic point of view.

Discussion of results. As a result of the workshop, a prototypical methodical procedure could be described which has proven to be successful: At the beginning there is a kind of praxeological analysis (in the sense of the Anthropological Theory of Didactics (ATD) (Chevallard, 1999)) of the task and the related mathematical domain. This focuses especially on the dialectic between technique (How?) and technology (Why?). This does not necessarily require the use of terms from the ATD. The central point here is to orient the didactic analysis of the material to the underlying questions that are answered by the specific piece of mathematics of the material. Of course, the necessary preknowledge (prerequisites), possible embeddings and references, possibilities of linkage as well as the desired knowledge should be taken into account, also with regard to its relevance. At this stage, the table for the task at hand could also be filled in. It can then be used to look for potential development needs or opportunities. In this respect it is useful to further differentiate questions which correspond to the task and the field of knowledge in a specific way. With regard to paths along which answers can potentially be developed by students, central techniques and technologies, as well as appropriate subtasks and partial solutions and the respective previous knowledge to be updated has to be worked out. In this process, it can be particularly useful to identify the semiotic and instrumental valence of used mathematical symbols and objects as possible activators of activities with regard to other objects, practices and concepts (Bosch & Chevallard, 1999). On this basis, the task can then be modified as desired and the table can be filled in again. The extension of the IBME profile can then be documented by comparing the initial with the modified table. With regard to support measures and feedback by lecturers, potential obstacles in the treatment of tasks by students should be considered with regard to necessary or possible solution steps and respective adequate support and its focus (content-related, strategic, methodological, material, impulse-giving). This leads over to didactic considerations in the narrower sense regarding the design of sequences of questions (possibly concept-maps or similar), suitable materials and media, teaching steps including their social forms as well as possibilities for diagnosis and feedback. Last but not least, expectations regarding the quality of the intended learning processes and their products has to be considered.

A final, albeit brief, look at the professionalisation workshop from the perspective of the three-layer model allows us to note the following observations: The planning and design of the workshop was done from the perspective of the third layer. Both the table and the examples presented for explanation have a theoretical background (here especially ATD, as indicated above), but this is not explicitly presented in the workshop itself. The material created against this background and the proposed procedure for modifying the tasks should be presented in such a way that they should not only be understandable and insightful for the participants of the workshop, but should also stimulate new processes of reflection regarding tasks brought along and their use in their teaching. The participants should thus be provided with new tools for processes on layer two. For this to work, it was helpful to develop an atmosphere of trust and a mutual recognition as experts. In this way, a community of inquiry could be created locally for the duration of the workshop and development could take place among both the participants and the organisers.

7.3.2. IBME Workshop in Agder, Norway.

Institutional context and goals. All newly appointed university lecturers in Norway are supposed to take a UNIPED (UNIversitets- og hogskole-PEDagogikk) course as a professional development course. The main objective of this course is to support a number of pedagogic skills required from a university lecturer (`www.uhr.no`):

- Plan and carry out teaching and supervision, both individually and in collaboration with colleagues, in a way that promotes student learning and professional development.
- Plan and implement R&D (research and development) based teaching and involve students in R&D-based learning processes.
- Select, motivate, and further develop appropriate learning activities and teaching and assessment methods in relation to academic goals and educational programs.
- Contribute to academic and pedagogical innovation through the choice of varied teaching methods that include the use of digital tools.
- Motivate personal views on learning and teaching reflecting the teacher's role.
- Analyse, prepare, and further develop course and program plans within lecturers' subject areas.
- Assess and document results from own teaching and supervision based on expectations in curricula and national frameworks for higher education.
- Collect and use feedback from students, colleagues, and society to further develop teaching and learning processes.
- Be familiar with relevant management documents related to teaching in higher education.

Norwegian universities offer UNIPED courses in different ways: for instance, the duration may vary from 100 hours to 150 hours extended over one or two semesters, the content depends on the priorities and resources of each university, but a typical UNIPED course focuses on innovative methods of teaching at university level. Usually these courses are not specific to mathematics teaching but encompass rather general topics of university education. Mathematics plays an important role in many students' study curricula and future careers; therefore, many university undergraduate and graduate programs contain mathematics courses. Teaching of mathematics focuses not only on the computational aspects but also on very important conceptual aspects which influence the choice of mathematical content, design of mathematical tasks and ways of communicating them. This brings the need to address the issues arising in relation to teaching and learning mathematics. In 2015, MatRIC, The Centre for Research, Innovation and Coordination of Mathematics Teaching at the University of Agder (UiA) in collaboration with the Norwegian University of Science and Technology (NTNU) launched a University Level Mathematics Teaching Course at NTNU complementing pedagogy courses offered by universities and university colleges. The course was designed to address the problems that university mathematics lecturers and students face: large classes (especially for engineering students), students' lack of motivation, diversity of students coming from different educational programs etc. Positive feedback from the participants indicated that such courses offer an opportunity for mathematics lecturers to grow professionally.

Organisation and contents. In order to introduce a wider audience to main ideas of inquiry-based mathematics education (IBME), PLATINUM organised a one-day workshop at the University of Agder in association with MatRIC. The workshop was organised with the following goals: to

- discuss the concept of IBME and describe the differences and commonalities in participants' views on what the IBME problem is and what it is not,
- present the IBME problems and their solutions,
- understand the ways how the non-IBME problem can become such, and to
- support and extend the network of IBME community of the University of Agder and include the participants from other Norwegian universities.

It was the second in a series of three workshops organised by PLATINUM as part of activities contributing to the Intellectual Output IO4 "Methods and Materials for Professional Development of Lecturers." The main activity of the workshop was the design, development, and piloting of activities through which (new) university teachers may be introduced to inquiry-based approach to teaching mathematics and gain insight into an inquiry-based task design, tasks structure, and their characteristics. The associated pedagogical and didactical ideas were also another focus of the event. Prospective participants who already used inquiry-based approach and tasks in their teaching were invited to bring own examples for the discussion at the workshop along with any tasks they could suggest for collaborative group discussions aimed at turning of "standard" tasks into "inquiry-oriented" ones.

In addition to local participants from UiA, Campus Kristiansand and Grimstad, colleagues from other Norwegian universities: NTNU, University of Stavanger, University of Trømso, Norwegian University of Life Sciences participated in the workshop. The event provided an opportunity to university lecturers who teach different mathematics courses in various study programs (engineering, teacher education etc.) to discuss their teaching practices together and share the experiences. The main activities of the workshop were:

- discussing the foundations of inquiry-based teaching and learning of mathematics,
- working together on mathematical problems selected from teaching units and tasks for student inquiry developed within the PLATINUM community, and
- discussing the foundations of inquiry-based teaching and learning of mathematics.

Reflecting on what has been experienced and how it can be implemented in own teaching practices, the activities in the workshop were organised in three main parts. For the first part, participants were split into small groups and offered a set of eight (proposed) inquiry-based mathematical tasks that were extracted from the contributions from several PLATINUM partners. The groups were asked to read and discuss some or all of the tasks. The purpose of the suggested discussion was to consider what makes a mathematics task an inquiry-based task. The following question were asked: What do we mean by an inquiry-based task? In what ways are they similar or different? How could you describe the inquiry nature of the task? What are the characteristics of an inquiry-based mathematical task? After the discussions in small groups all participants were invited to the general discussion in which the characteristics of an inquiry-based mathematical task were suggested. The characteristics reflected the discussion about the proposed tasks and included accessibility, openness, openness to multiple strategies, opportunity to iterate, motivation to investigate and explore, evaluation, necessity of reasoning, possibility to generalisation and specialisation, suitability for group discussion. The moderator gave the comments on the discussion pointing out that in a mathematical course it is necessary to consider the place for the following: exposition, investigation, exploration, focusing, evoking, stimulating, motivating, and problem solving.

In the second part of the workshop participants worked on inquiry-based tasks. They were asked to choose one of three options: (1) try to design your own tasks; (2) talk with others about the tasks you have brought yourselves; (3) work on the Linear Algebra or Analysis (or other) tasks, modifying them to be more inquiry-based. For Option (3), a folder was provided containing sample sets of tasks from regular mathematics courses taught at the University of Agder and at Loughborough University. These were tasks used by colleagues with their students; some were intended as inquiry-based tasks and some were not designed to be inquiry-based. Four groups were formed, one to work on Option (1), two to work on Option (2) and one to work on Option (3). The groups worked together for 45 minutes and then each group presented their work to all participants. One of the groups suggested the design of the task that could be offered to the students of a teacher training program, namely asking the students to compose an inquiry-based task.

Example of task (1). Working in groups of two or three, develop a series of realistic mathematical problems that you can meet in real life either in collective life or in your personal life. These problems should require one to create and arrange a series of quantitative information into rectangular arrangements (RA) in $n \times m$ dimensions. Both n and m should be equal to or greater than 3. Then, pose some realistic problems that can be answered by at least two of the following operations on these RAs: multiplication, finding an inverse, addition, subtraction, and multiples of the RAs. Find the answers. Another group discussed the problem brought to the workshop by one of the participants who reported that it had been tested at school.

Example of task (2) (unfolding a three-dimensional cube) Suppose that you have a cube. How many unique ways are there to unfold it? (Or how many different nets of a three-dimensional cube can be obtained?) The answer to this combinatorial problem is eleven, but this problem can be modified in various ways. The group discussed a possible modification: ask for the minimal number of adhesive flaps to be included in a net so that it can be folded and glued together to make a waterproof cube. Another possible modification is the transition from a cube to a dice. One can ask the question how many different possibilities there are for the distribution of the natural numbers ranging from one to six on the sides of a cube, that make it a dice. In a classical setting, a dice possesses the property that the sum of the natural numbers on two opposite sides is always equal to seven, but this limitation still leaves two different possibilities or orientations. This difference becomes indeed visible when comparing a European dice with an Asian one. However, this classical constraint can also be omitted. Some tasks using simulations (generated by programming tool SimReal[3] were presented. Examples included Pythagoras Theorem, exponential functions, Green's theorem, and complex numbers. Finally, in the third part, participants reflected on their experience and were asked to rate the interest and usefulness of the workshop, commenting on what was good about the day and suggesting the ways for possible improvement of future events. The survey was distributed and the responses were analysed. The reflections showed that there were discussions on

- the degree of openness on IBME-problems;
- balance of the learning goals and syllabus;
- the interplay between the problems and other classroom factors such as students' group discussions;

[3] `https://grimstad.uia.no/perhh/phh/MatRIC/SimReal/Menu/Science.htm`

- teacher's role in guiding and supporting students' discussions, explorations, and reasoning;
- and teacher's role in bringing students' small group work into a whole-class discussion.

One of the participants wrote: "It was very nice with these discussions. The only thing I missed, are more examples of good inquiry-based tasks. I especially need more examples of good tasks at undergraduate level." According to the PLATINUM proposal, the goal of the project is the creation of community of inquiry, and the activities on the IBME Workshop in Agder contribute to this goal in a significant way.

Discussion of results. The experiences gained by the workshop participants were reflected in the discussions of the characteristics of IBME tasks and the ways how good IBME tasks can be designed. Participants focused their attention on the goals of such tasks that should address a number of important issues: to motivate students to foster curiosity, to ask questions, to explore topics, to be engaged in active learning. The participants argued that since the IBME is based on problem-solving, such tasks must be specifically designed to support the development of deep understanding of the material and intellectual problem-solving skills. Students must be able to build up on their previous knowledge which requires an accurate assessment of such knowledge by the teacher. Indeed, the problem that can be classified as an IBME task for one group of students may not be such for another group. The level of inquiry should be also taken into consideration: there are different levels of inquiry with respect to openness, from very structured to fully open. For example, the problem in which the result is known and students need to choose and apply the correct rule or method would be of limited inquiry while the problem which is set up by students themselves and solved by using the knowledge from different areas to develop the procedure would be of high level of inquiry. To achieve the goal the teacher should be able to make the right choice of the level of inquiry with respect to the level of mathematical content and to the level of students' prior knowledge. The question was asked: How much can be removed from a student's 'agency' before a task ceases to be an inquiry task? Along with the task design the participants indicated the importance of class management in the IBME process. They named some important topics that can contribute to successful learning, such as students' collaborative study, the crucial role of students' freedom to choose the tools and direction of solving the task. The participants also attempted to define the relationship between exploration tasks, modelling tasks and inquiry tasks. They pointed out that not every modelling task is an exploration task and not every exploration task is an inquiry task. But in general there were different suggestions to define what each of these tasks is. It seems that there is no agreement in such definitions and it depends not only on the content but also on the background and preparation level of students.

7.3.3. IBME Workshop in Madrid, Spain.

Institutional context and goals. The Faculty of Mathematical Sciences has been contributing to teacher professional development for more than 60 years. Since 2007, the Extraordinary Chair UCM Miguel de Guzmán, taking into account this trajectory, has contributed with programs of continuous development of university lecturers, developing formative actions of post-graduates in Mathematical Education at local, national, and international level (Corrales & Gómez-Chacón, 2011).[4] Concerning the professional development of the mathematics lecturers, this is a great milestone in the

[4] `http://blogs.mat.ucm.es/catedramdeguzman/proyecto-novelmat/`

country. It is an exceptional case because professional development for university lecturers is usually considered as a generalised pedagogical training and it is not mediated by specific mathematical knowledge.

In our tradition in the implementation of the teacher's professional development, research and development are inseparable. In the PLATINUM Project, the development of this perspective is based on the three-layer model presented in Chapter 2 of this book (see also, Jaworski, 2019). When we want to promote teaching through inquiry-based pedagogy we consider mathematics as "a process and an experience." Knowing mathematics is equated with doing mathematics. In our group, research in mathematics education has focused on examining the characteristics of the context in which this "doing" is fostered. Designing research-based tasks requires deep analysis of the "mathematical experience" that is generated in both students and lecturers. Thus, the creation of inquiry-based activity for the students is itself an inquiry process: lecturers learn from the practices resulting from their teaching designs. In the UCM case study presented in Chapter 16, it is suggested that the research strategy followed in the professional development for the teacher is framed within Design-Based Research. The design research is characterised mainly in terms of simultaneously developing theory and designing instruction; engaging in iterative cycles of design, enactment, analysis, and revision; and performing fine-grained analyses to link processes of enactment with outcomes of interest.

Organisation and contents. In what follows we will focus on the Course proposal developed within the framework of PLATINUM. This was entitled: "Inquiry-based education in mathematics at university level: Examples for the classroom." Aims of the course are the following:

- initiation in the Inquiry-based learning approach applied to teaching and learning situations at university level;
- development of methodological skills to design inquiry tasks, and
- knowledge of resources,i.e., examples of inquiry-based projects in university mathematics teaching.

It lasted ten hours distributed in two specific and complementary parts. Each part forms a unit by itself.

First part: A workshop for five hours addressed to senior and novice lecturers. Forty-five participants attended. The group was mixed from five universities: university lecturers teaching in mathematics in the faculties of Science (Mathematics, Physics, Computer Science), engineering faculties, and faculty of Education; research assistants (PhD students in the last year who collaborate in teaching or planning to teach soon); and students in Master's degree programmes in teacher training. The contents were the following:

(1) Inquiry approach talk: Teaching and learning mathematics through inquiry.
(2) What do we mean by an inquiry-based task?
(3) Inquiry projects and tasks implemented in the classroom in Spain by the Spanish PLATINUM Group at UCM.
    (a) Escape room. Elements of ordinary differential equations;
    (b) Teaching linear algebra and video games. Affine transformations and rigid motion;
    (c) Matrix factorisation. Numerical methods (see Chapter 16).

Second part: Only addressed to those participants of the first part who are research assistant in the UCM (PhD students in the last year who collaborate in teaching or planning to teach soon). For five hours, 6 participants attended. The aim and contents of this part are to deepen different aspects of the first part and design a task or project using the inquiry approach. Taking into account the contents of the first part, this part focuses on:

- Encourage self-reflection on the teaching of the novice lecturers after receiving training in Inquiry based approach.
- Improve the knowledge of professional development reflection about level of inquiry and progressive movement to abstraction and symbolisation in affine transformations and rigid motions.

Discussion of results. Two experiences are highlighted about the characteristics of an inquiry-based mathematical task: meaning of an inquiry-based task and example of complex tasks through an inquiry project.

1. What do we mean by an inquiry-based task?

For this aim a set of eight inquiry-based mathematical tasks was prepared. This sample sets of tasks come from regular mathematics courses taught by several PLATINUM colleagues with their students; some were intended as inquiry-based tasks and some were not designed to be inquiry-based. These tasks were discussed at the workshop in Brno for the PLATINUM team and also at the workshop in Norway. Our purpose in discussing these tasks is to consider what makes any mathematics task an inquiry-based task. At the UCM Workshop the groups (composed of 5 participants) worked together for 45 minutes and then each group presented their work to all participants. After the discussions in small groups all participants were invited to the general discussion in which the characteristics of an inquiry-based mathematical task were suggested (45 minutes). At the beginning of the analysis the proposed tasks were characterised. Characteristics such as openness, motivation to investigate and explore, autonomy of students, evaluation, necessity of advance reasoning, possibility to generalisation and specialisation, suitability for group discussion. In relation to the recognition of tasks and their use in the Spanish context there were tasks whose level of knowledge was very elementary, not corresponding to the university level. This raised discussion about transferability, learning outcomes and the kind of purposes and criteria according the local syllabus. The discussion allowed us to ascertain the range of inquiries according to the profiles of students and curricula in each country and in relation to the phases of inquiry were focused on regulation and inquiry pathways in the sequences of actions by the lecturers.

Also, and not less important to consider is the conceptual model of the inquiry process. We see inquiry as having both epistemological and strategic aspects, with developments on the two fronts reinforcing one another. The epistemological point of view of mathematical knowledge is the key. The inquiry approach to mathematics is not only a pedagogical strategy. It seems pertinent to take into account some of the tensions that seem inherent and at the same time can provide us with tools for teaching. These are tensions between the development of the mind's research habits and the progression of mathematical knowledge with the necessary attention to curricular progression, the tension between the internal and external sources of mathematical activities and differences between inquiry paradigms according the mathematical field (Geometry, Calculus, etc.). In the next section we describe an example of how to make this articulation developed together with the new lecturers.

2. Design of inquiry pathways with complex tasks.

We describe the Inquiry project "Teaching linear algebra and video games." Affine transformations and rigid motions" as a professional development reference situation developed together with the participants in the workshop (novice teachers), i.e., as a reference situation that can be developed with their future students. Below we present the project's tasks, key elements in the process of inquiry, and didactical and mathematical moments.

We take the game SILENT HUNTER III as a starting point. Silent Hunter III is a submarine combat simulator for PC developed by the company Ubisoft and published in March 2005, set in World War II. If you have never played the video game, you can watch the YouTube video `www.youtube.com/watch?v=2Xa4gWCHlFU` to familiarise yourself with the game. Some of the missions that take place in the video game use one of the most significant elements of a submarine, the periscope (see Figure 7.2). Using this tool, players are able to inspect a large part of the map, view enemy positions or establish safe routes for the submarine's travel.

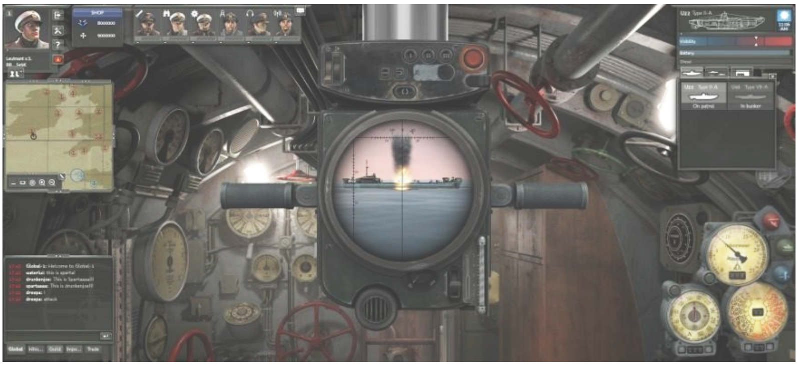

FIGURE 7.2. Periscope of a submarine of the video game SILENT HUNTER III

Task 1. What kind of mathematical tasks/activities would you propose to a first-year student of the degree in Mathematical Engineering for teaching the contents of Mathematical Elements or Linear Algebra? Describe some of them.

Task 2. With the help of dynamic software GEOGEBRA, create a 'hypothetically ideal' periscope that can move up and down and rotate a maximum of 45 degrees, from a starting point to any of the cardinal points. Prior to using GEOGEBRA to solve the problem, write down your thought process, noting steps you would take, necessary mathematical knowledge, and places where you struggled.

Task 3. If you had difficulties completing the GEOGEBRA construction, specify the causes: mathematical knowledge, instrumental construction with GEOGEBRA, etc.

Task 4. Based on the activities carried out, answer the following questions:

(a) Provide the matrix representations for the following transformations:
   - (i) The z-axis rotation with angle $\alpha = \frac{\pi}{4}$;
   - (ii) the translation of vector $\vec{v}$;
   - (iii) the axis rotation the line r and angle β, $\beta = \frac{\pi}{6}$ (the axis rotation by angle β around the line r)

   What effect does each of them have on the periscope?

(b) Is it the same to turn the periscope 45° to the west and then to raise it 1m than to raise it 1m and then rotate it 45° to the west? How did you work this out? What is the name of the movement resulting from applying both?

(c) Can you establish the set of fixed points for each of the transformations in the first question? Did you work this out intuitively by using the GEOGEBRA construction, or by applying the definition?

(d) In what position could an enemy find a place to hide from the radar of the periscope, assuming that the periscope is located in the starting position and can only turn in an east-west direction?

Key elements in the process of inquiry. Participant lecturers worked with these specific linear algebra tasks, identified key elements in the process of inquiry, and paid attention to how to formulate questions such that they encourage students to move between the embodied, symbolic and formal worlds of mathematical thinking.

Multiple modes of thinking which result in richer conceptual understanding of the concept of affine transformation (linear maps) and rigid motion are explored. In order to acquire experience in inquiry-based teaching, it will often require moving between the levels in the same lesson. It might give more student responsibility in open inquiry, while, at the same time, it could be required support through structured inquiry. For lecturers, these challenges included: (a) the grading of inquiry levels of concept use, technique use, and teacher regulation of activity; and (b) the relationship of students' intuitive, informal, or flowering ideas to conventional and more formal mathematics. Figure 7.3 shows a template that helped the participants discuss the determination of sequences of inquiry pathways, concept visualisation, formalisation, and symbolisation.

Level of inquiry Linear Algebra	**Profile of mathematical thinking difficulties**	**Phases of inquiry**				
		Questions and observations	Regulation	Inquiry pathways	Results of inquiry: concept visualisation	Results of inquiry: formalisation and symbolisation
Structured inquiry						
Guided inquiry						
Open inquiry						

FIGURE 7.3. Levels of inquiry and intuitive and formal thinking

Finally, the implementations developed in the classroom were shared. As indicated above we conceived the inquiry as having both epistemological and strategic aspects. For a specific characterisation of the area of knowledge in which the inquiry takes place the inquiry project was described in didactic terms following different moments of mathematical organisation: initial, exploratory, work of the technique and concepts, technological-theoretical, institutional, of engagement, of critical alignment and of motivation and, articulated around tasks associated to concrete objectives.

Some challenges in order to support professional development of lecturers were raised: what do we mean by an inquiry-based task in mathematics, what is specific to inquiry in mathematical work and differentiates it from another knowledge. The development of rich tasks contributes to mathematicians' feelings of effectiveness in their discipline. The workshop (with its two parts) addressed a number of other ways in which lecturers can integrate IBME approach. These include activities such as choosing problems, predicting student reasoning, generating and directing discussion, asking for questions that extend student knowledge, guiding students for high-quality explanations and mathematical justification.

We see the epistemological point of view of mathematical knowledge and relation of epistemological and cognitive dimension of inquiry process are the key attributes. Two aspects were highlighted in order to make progress in the professional development of participants in the determination of sequences of inquiry pathways of the tasks:

- The relationship of students' intuitive, informal, or flowering ideas to conventional and more formal mathematics
- The nature of the question (extra-mathematical system and mathematical system).

The nature of the question, obviously, has consequences for the inquiry process. In the case of questions that come from an external source, such as is the example of video games project, their transformation into questions of a mathematical nature is an important part of the inquiry process, which involves a modelling process. Mathematical inquiry, when it comes from external situations, necessarily includes a modelling process and combines several logics. In the shared experience, a greater depth in the relations and interactions between the two systems is open: an extra-mathematical system and a mathematical system. Each one has its own logic and, consequently, it is necessary to differentiate epistemological aspects as well as strategy aspects of the inquiry process.

7.4. Summary and Looking Ahead

The experience of the three workshops provided us with valuable information and allowed us to highlight milestones and challenges in addressing an international workshop offered to academic staff. Here we highlight two challenges by answering the following questions:

- What do we mean by an inquiry-based task?
- How can I support teachers' growth on inquiry-based teaching approach?

Regarding the meaning of inquiry tasks, the shared experience of analysis of the same set of tasks in two different workshops (Norway, Spain) showed that there exist varied conceptions of 'inquiry' by the lecturer's participants depending on their mathematics views and according to the content of the discipline they teach. Therefore, it seems pertinent to pay attention to the conceptual model of the inquiry process that we bring to this work and the role of this model in the sequence of activities that we employ. We see inquiry as having both epistemological and strategic aspects, with developments on the two fronts reinforcing one another. A lecturer should think about the forms of knowledge and procedures in mathematics when structuring questions of the inquiry tasks in order to let his/her students engage in authentic and productive inquiry. When undertaking inquiry students are motivated in the sense that the process is driven by an explicit intention to find out. In each of the three different workshops (Germany, Norway and Spain) examples have been worked on that favour this epistemological analysis of knowledge.

This epistemological analysis of knowledge helped to characterise the knowledge domain in which the inquiry is developed (it is different if we are working on geometry or algebra or analysis contents). The knowledge domains have their specificity and address determined aspects of mathematical thinking. This will require a learning accompaniment with adequate didactic elements.

The respective specification of these aspects cannot be made a priori in advance of a workshop, but instead results in particular from the exchange between the participants. This is not only due to the great epistemological diversity of mathematical content or pedagogical-strategic possibilities for action, but also to the specificity of each IBME development process. This will be briefly illustrated again in the following against the background of our experiences using the three-layer model.

Certainly, the students' learning should be optimised and deepened. Therefore, the first layer orients itself towards a common goal. However, what this means in the respective context, thus also with regard to the possibilities and limits of learning activities, can be very different. The same learning activity can mean a big step towards a greater taking of responsibility and independence in one context, but in another context it can essentially mean the unreflective reproduction of something previously trained. Explicating and thus disclosing the respective didactic contract (Brousseau, 2002) is therefore also of great importance. Decisions based on this contract significantly determine the activities and their assessment at layer 2 of the three-layer model presented in Chapter 2. Here, too, it is not possible to say absolutely and without taking into account the teaching-learning culture prevailing at a university what teaching activities promote or hinder IBME in the respective subject context, etc. (see layer 1 of the three-layer model).

The respective workshops intended to establish a community in the sense of Layer 3 of the three-layer model. This means in particular to enable a trusting exchange. It seems trivial, but without a personal commitment, without questioning one's own assessments and experiences (see also Layer 2), an intentional development in the sense of IBME is impossible. The introduction of didactic terms and concepts and the respective focus on mathematical aspects at all three workshops allowed for an open discussion and reflection by objectifying the exchange to a certain extent. The degree to which prior clarification is necessary or useful depends, of course, on the context of the workshop and the previous knowledge and interests of the participants. What is already clear, however, is that a too strong institutionalisation, possibly linked to a grading in the sense of "IBME lecturer of grade XY," could be quite problematic, as this would create an assymetrisation that could stand in the way of the necessary or at least helpful psychological group processes. Lecturers might then behave in a similar way to many students, trying to undermine the fulfilment of requirements, etc.

A limit of the workshops carried out became clear in the respect that layers 1 and 2 are in a certain sense only presented virtually. However, whether further developments of tasks, their use, and so on, modify the didactic contract or lead to the intended mathematical learning actions had to be left open here. Clarifying this in exchange with the students represents, in a sense, the objectification step that cannot be achieved in this way, but which is immanently linked to IBME and is reflected in the necessary inter-connectedness of the three layers. A workshop detached from such clarification steps can therefore only be a step to initiate processes. In the IBME sense, it is an important step, but it would be best to make it an ongoing activity, also to enable further group processes and a deepening of the discussion and reflection on mathematical teaching.

Finally, we outline below three aspects of progress.

1. How is the progress related to issues in mathematics professional development? Professional development is a very broad field that also addresses so-called personality development. The workshops focused on professional aspects and the provision of tools (e.g., Table 7.1, Table of dimensions for IBME task, and Figure 7.3, Levels of inquiry and intuitive and formal thinking) with which academics can develop subject-related teaching. The tools have similarities and differences, depending on whether there is a stronger focus on the subject logic or on the process logic. All workshops showed how tools can be used and offered opportunities to do so. As far as we can see, the respective institutional teaching-learning contexts were always taken into account. In this way, local and, if necessary, small steps should be made possible. The expertise of the participants was used. In local contexts, the collaboration between the mathematics teachers and mathematics educators is a crucial point. This is not something that is specific for the university education community. Workshops provide opportunities for mathematicians and mathematics educators to discuss the problems of education at the university level, the pedagogy of teaching, the development of innovative teaching methods etc.

2. How is the relation of the workshops to inquiry-based practice and inquiry communities? All workshops were organised in such a way that for a short period of time CoIs have been created, in which conversations and discussions could take place on equal terms. The focus was more on the innermost layer, the development of tasks design, hence, on the use of the tools and possible products in the form of further developed tasks. In general, the establishment of a kind of CoI might be seen as a prerequisite for fruitful discussions. It is also noteworthy that another level of inquiry community occurred (at least in Norway and Spain) as the workshops are aimed at contributing to the development of CoI at the national level, promoting the collaboration between different universities.

3. How are the consequences for developments in teaching and students' mathematical learning? The development in teaching and mathematical learning was connected to the main goals of the workshop: the participants presented their tasks, debated on how to transform them to inquiry-based tasks and discussed what the inquiry-based task in the specific teaching situation can be. The concrete tasks made possible to experience that a certain systematic approach to tasks and their design can be helpful. The workshops were organised in such a way that the participants could connect with their professional competence, their teaching objectives, etc. One challenge that became evident for those who run these workshops is their own professional competence. The imagination must be stimulated in the relevant professional context for the participants.

7.5. Conclusions

With the experiences of the different workshops (Germany, Norway, and Spain) we have tried to highlight key elements for an effective teaching of the inquiry approach at university level. These materials are proposed from three local communities that take into account institutional and theoretical frameworks to support the professional development of lectures through teaching, research and participation in learning communities or communities of inquiry. The current understanding of university professional development for lecturer has some institutional constraints or tensions. With the examples proposed here, a design-based community that integrates research and participation in a learning community can better facility lecturer learning. From PLATINUM project's study and experience, we note several requirements for developing a design-based community. First, develop a strategy for linking research and practice perspectives in the programme. In the three-layer model we suggest a layered structure, but the big challenge is to operationalise design tools in the interaction of these layers. Second, to engage lecturers in designing instructional tasks and to detect their mathematical and pedagogical challenges. In order to facilitate lecturers' task designing, the workshops have proposed some starting points. Third, develop of strategies to allow teachers to incorporate theoretical ideas into their instructional task design. The adoption of the three above considerations could be taken into account for the design of a workshop applicable in various contexts.

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