

CHAPTER 6

Creating Teaching Units for Student Inquiry

ANDRÉ HECK, LUKÁŠ MÁŠILKO

6.1. Introduction

Tasks determine to a large extent how students develop mathematical thinking abilities and become fluent in applying mathematical methods and techniques. As Stein et al. (1996, p. 459) put it: “The mathematical tasks with which students become engaged determine not only what substance they learn but also how they come to think about, develop, use, and make sense of mathematics.” They distinguish (p. 466) in mathematical tasks four increasing cognitive demands: (1) memorisation; (2) use of formulas, algorithms, or procedures without attention to concepts, understanding, or meaning; (3) use of formulas, algorithms, or procedures with connection to concepts, contexts, understanding, or meaning; and (4) “doing mathematics,” including complex mathematical thinking and reasoning activities such as making and testing conjectures, framing problems, looking for patterns, and so on. Tasks at the highest level of cognitive demand are complex, possibly ill-structured, and require students to make strategic decisions, make connections between concepts and contexts, reason in a mathematical way, and explain their thinking. In other words, student are invited to work as a mathematician or as a professional using mathematics in her/his field.

A good and varied selection of academic tasks is especially important at university level, where the expectation is for students to spend considerable time outside of class studying course material and doing homework. We refer to familiar or routine tasks that aim to increase student fluency with mathematical content and techniques as tasks that promote procedural understanding of mathematics. When procedures are used with connection to concepts, contexts, understanding, or meaning, or when tasks encourage doing mathematics, then we speak of tasks focusing on conceptual understanding of mathematics. In mathematics education at university level, especially in service teaching,¹ student tasks and activities are in practice more often directed towards procedural understanding than to conceptual understanding of mathematics and use of higher-order thinking skills (see, for example, Artigue et al., 2007). The main goal of the PLATINUM project was to explore possibilities to shift the balance in student learning towards conceptual understanding of mathematics. As part of their inquiry at all levels of the three-layer model explained in Chapter 2, PLATINUM partners formed communities of inquiry to develop teaching units for student inquiry, to try them out in their own practice, to evaluate the use of these units, and to document their work. In this chapter we report on this work (Intellectual Output 3 of the project; see Section 2.5), put it in a broader perspective of inquiry

¹Service teaching of mathematics is an umbrella term for teaching mathematics in higher education outside mathematics programmes, e.g., teaching mathematics to engineers, students in life sciences, etc.

into mathematics education, discuss the possible role of ICT in student inquiry, identify from the developed teaching units some general task features and guiding design principles (including those for students with identified needs), and discuss what a community of inquiry can achieve in practice.

6.2. Frameworks Used in PLATINUM for Designing Student Inquiry

Much research has been done focusing on task design in mathematics education. The ICMI study 22 (Watson & Ohtani, 2015) is a very good source of information. We distinguish three types of frameworks that are used to underpin the task design for student inquiry:

- a grand theoretical frame (e.g., constructivism) or an intermediate-level frame such as Realistic Mathematics Education (RME) (van den Heuvel & van Zanten, 2020), the Theory of Didactical Situations (TDS) (Brousseau, 2002), the Anthropological Theory of the Didactic (ATD) (Bosch et al., 2019), and Commognitive Theory (Sfard, 2008), to name a few;
- a learning cycle instructional model for mathematics and science built upon general notions about how people learn (Bransford et al., 2000; Donovan & Bransford, 2005), such as the 5E-instructional model of Bybee et al. (2006);
- a model of processes and actions in professional practice. Examples of this type of framework are a list of words denoting processes and actions when mathematicians pose and solve problems (Mason, 2008), a categorisation of tasks that encourage concept development, an identification of design principles that make teaching for conceptual understanding more effective (Swan, 2008), a mathematical questions taxonomy (Smith et al., 1996; Pointon & Sangwin, 2003), and a modelling cycle (e.g., Blum & Leiß, 2007; Heck, 2012).

We elaborate on some of these frameworks and discuss how they played a role in the design of teaching units by PLATINUM partners. Most of our attention is on the third type of frameworks because they seem, in our view, closer to the world of university lecturers and more appealing to them.

6.2.1. Use of Intermediate-Level Frames in PLATINUM. Many mathematics education researchers use an intermediate-level frame such as ATD or RME to position their developmental work. For example, partners from the Leibniz University Hannover (LUH) use concepts from ATD, TDS, and Critical Psychology (Holzkamp, 1995, 2013) in their case study presented in Chapter 14 to analyse their teaching and learning practice. Partners from the Borys Grinchenko Kyiv University (BGKU) refer to RME principles when they describe in Section 8.4.1 their view on mathematical modelling. Partners from the University of Amsterdam (UvA) also discuss the attractiveness of RME principles in their case study on teaching Systems Biology (see Chapter 12), in particular the idea to routinely invite students to explain and justify their mathematical thinking, their solution strategies, and actions in open-ended activities. Yet they did not adopt (and explain why not) the RME-based inquiry oriented instructional approach of Rasmussen and colleagues (Rasmussen & Kwon, 2007), in which emphasis is on student reinvention of mathematical concepts, on student inscriptions and their role in the development of the mathematics, and on instructor inquiry into student thinking.

Kuster et al. (2018) identified the following most important principles of inquiry-oriented mathematics instruction: (1) generating student ways of reasoning, (2) building on student contributions, (3) developing a shared understanding, and (4) connecting to standard mathematical language and notation. Kuster et al. (2019) developed

an instrument for scoring a lesson along seven inquiry-oriented instructional practices. Similarly, spidercharts have been developed as instruments within the PLATINUM project (see Chapter 3) to facilitate project-wide thinking and communication about activities in local communities and to promote reflection on and further elaboration of a common vision on inquiry-based mathematics teaching and learning from the student perspective, the teacher perspective, and the community of inquiry perspective. The spidercharts in PLATINUM are not a scoring tool, but a reflection tool for a community of inquiry. For example, while working on a basic mathematics module for first-year students in biomedical sciences, the UvA community of inquiry was supported by the spidercharts in the pedagogical decision-making processes and in discussions about suitability of RME for its design of student inquiry. At first sight it seems attractive to use instructional activities in which students reinvent the concept of direction field (Rasmussen & Marrongelle, 2006), Euler's method (Kwon, 2003), solution of linear systems of ODEs (Rasmussen & Blumenfeld, 2007), or bifurcation diagram (Rasmussen et al., 2019; Goodchild et al., 2021), and in which students more or less constitute the formal mathematics. But in the reality of university teaching, the UvA partners had pragmatic reasons for rejecting the principles of guided reinvention and emergent modelling in their mathematics module: limited student-teacher contact time, insufficient availability of suitable learning spaces for small-group work, large number of students and their differing mathematics background that would complicate the reinvention and emergent modelling processes, and a mismatch with the dominant teaching and learning culture in the discipline. Another obstacle foreseen by UvA partners in their course design was the extent to which lecturers could elicit and inquire into student thinking in practice, which is considered a crucial aspect of design research and inquiry-oriented education. Guidance and monitoring small-group work and utilising student work to promote a shared and more sophisticated understanding of mathematics commensurate with the important mathematics concepts and conventions addressed in the module was cumbersome given the layout of the tutorial rooms and the number of students present during practice sessions.

The above objections and hesitations toward the inquiry-oriented instructional approach can also be found amongst university lecturers toward other inquiry-based approaches that are based on a grand theoretical or intermediate-level frame, for example grounded on (socio-)constructivism or cultural-historical activity theory, especially amongst lecturers involved in service teaching of mathematics. Often these lecturers feel uncomfortable with a constructivist perspective on mathematical representations and acts of representing. From a constructivist point of view (see, for example, Cobb et al., 1992; Davis et al., 1990; von Glaserfeld, 1995; Jaworski, 1994), the learning of mathematical representations should not take place in a transmission approach of instruction, in which lecturers explain for their students the meaning of mathematical and scientific representations and how they are to be used. Instead, informal representations created and used by individual learners during the learning process should play a role in the route towards conventional mathematical notations. In other words, in a constructivist perspective, both acts of representing and representations are a means of constructing mathematical knowledge and understanding by students. University lecturers involved in service teaching often feel that there is too little space in the already overlaid mathematics courses for a constructivist approach, which is more time-consuming than traditional instruction. Often they do not have the power to reduce the mathematical content of the courses in order to make space for student inquiry activities, or they lack personal experience with a constructivist approach. They may even have a limited view of grand theoretical and intermediate-level frames, and

are unaware that these frames leave space for various approaches to student inquiry with respect to intellectual sophistication and to student participation or locus of control (cf., Wenning, 2005, in the context of science education).

In PLATINUM we distinguish student inquiry on the following levels, ranging from rather closed to completely open inquiry work:

- *limited inquiry*, in which students follow directions and make sure that their results match the requirements set in advance;
- *structured inquiry*, with no predetermined answers but conclusions solely based on students' investigation;
- *guided inquiry*, with no predetermined method but students having to determine how to investigate the problem;
- *open inquiry*, with no predetermined questions but students proposing and pursuing their own questions;

Under the assumption that the sum of the levels of teacher and student participation in each of the above inquiry types is roughly the same, the above types of student inquiry are ordered with increasing student participation and independence (locus of control) and with decreasing degree of teacher's guidance. Many lecturers are willing to move students in a course from a teacher-dependent to a more teacher-free and independent role, i.e., to shift the locus of control gradually from teacher to student, but are afraid that a course is too short for doing this. Promotion of inquiry-based mathematics education often boils down to breaking barriers like the ones mentioned.

6.2.2. Use of Learning Cycle Models in PLATINUM. PLATINUM partners have also used learning cycle models, not only to design their student activities but also to analyse how these activities actually went in classroom practice. For example, to compare the design of their inquiry task with the student actions in class, partners from the Brno University of Technology (BUT) refer in Section 8.4.2 to the model of Pedaste et al. (2015) for IBME activities, consisting of the phases Orientation—Conceptualisation—Investigation—Conclusion—Discussion, and to a simple 4-stage modelling cycle, consisting of Understanding task—Establishing model—Using mathematics—Explaining results.

Quite popular in the PLATINUM project has also been the 5E-instructional model of Bybee et al. (2006) and the 7E-model of Eisenkraft (2003) for characterising tasks in a teaching unit for student inquiry.² The 5E-model requires instruction to include the following phases: engage, explore, explain, elaborate, and evaluate. The UvA partners have, for example, used the 5E-model to characterise the task sequence about enzymatic kinetics developed in their case study presented in Chapter 12 (see Table 12.1, p. 224). The 7E-model expands the Engage phase into two components—Elicit and Engage. Similarly, the 7E model expands the two phases of Elaborate and Evaluate into three components—Elaborate, Evaluate, and Extend. Partners of the Complutense University of Madrid (UCM) have used the 7E model in their documentation of the teaching unit about matrix factorisation, which is part of the case study presented in Chapter 16, to typify the student activities (see Table 6.1).

The above examples illustrate that a learning cycle model not only provides lecturers and educators with a documentation and assessment tool that they could use over time to both tell the story of their teaching and the learning of their students in a particular setting, but also supports lecturers and educators in developing learning experiences for their students. The latter use of a learning cycle model fits very well

²Maybe the popularity of the 5E- and 7E-model originates from the inclusion of these models in a working document used in the PLATINUM project to help partners document their work.

<i>Part</i>	<i>Activity</i>	<i>E-Emphasis</i>
Task 1	Apply the Gaussian elimination method to compute the LU factorisation of matrices in a special class of matrices	Elicit
Task 1	Propose a conjecture	Engage
Task 2	Check the conjecture: does it hold for other classes of matrices	Explore
Task 2	Modify the conjecture	Engage
Task 3	Find a general result	Explore
Discussion	Each group makes their results public	Explain
Discussion	Groups assess, accept or criticise the results and opinions of other groups	Elaborate, Extend

TABLE 6.1. Characterisation of student activities in the matrix factorisation teaching unit of the UCM partners, presented in Chapter 16, via the 7E-instruction model of Eisenkraft (2003).

with the design of structured or guided inquiry. Then an activity sequence contains all phases in the learning cycle model, which are (repeatedly) divided over the activities. Use of a *cyclic* instructional model is intentional because it emphasises the role of the model as a formative documentation and assessment instrument that supports lecturers in designing learning experiences for their students by reflecting on where their students have been, what they have learned, and what they might do next. It also reflects that student inquiry is ideally a cyclic process with more than one iteration.

6.2.3. Using Models of Processes and Actions in Professional Practice.

There are many different definitions and interpretations of the term inquiry-based mathematics education (IBME). All university lecturers have an intuitive feel for what is meant by this term and whether a clear definition is given or not, they probably recognise inquiry-based teaching and learning of mathematics. This is especially true for the following conceptualisation of IBME formulated by Dorier and Maaß (2020), in which active engagement of students with mathematics is central:

Inquiry-based mathematics education (IBME) often refers to a student-centred paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, look for mathematical and scientific ways of how to answer these questions (like carrying out experiments, systematically controlling variables, drawing diagrams, calculating, looking for patterns and relationships, and making conjectures and generalisations), interpret and evaluate their solutions, and communicate and discuss their solutions effectively.

What does *active engagement* with mathematics at university level mean? Mason (2002) provides a list of words he believes to denote processes and actions when mathematicians pose and solve mathematical problems. He distinguishes the following innate powers employed by practitioners when they work on mathematics: “exemplifying, specialising, completing, deleting, correcting, comparing, sorting, organising, changing, varying, reversing, altering, generalising, conjecturing, explaining, justifying, verifying, convincing, refuting, and depicting” (p. 125). Mason (2002), Mason et al. (2010), and Mason and Johnston-Wilder (2006) discuss in detail how

the design of student tasks might benefit from using these words to give students a richer experience of aspects of mathematical thinking. The cited authors are of opinion that mathematical thinking tasks should be rich tasks that encourage students to be assertive and active rather than taking a passive approach to learning: tasks should enable students to encounter significant mathematical ideas and concepts and to discuss them with peers, allow them to use their ‘natural’ thinking powers to work on mathematics, involve various ways of thinking, engage them in the use of precise mathematical language, and be appropriately challenging. The listed processes and actions taken from mathematical practice are expected to help designers of rich mathematical tasks. Mason and colleagues provide guidance to underpin this expectation in the form of a variety of tactics. An example of how this guidance helps in practice can be found in the paper of Breen and O’Shea (2018) in which they select the following six types of tasks that would engage students in the practices and habits of minds of research mathematicians: (1) generating examples, (2) analysing reasoning, (3) evaluating mathematical statements, (4) conjecturing and/or generalising, (5) visualising, and (6) using definitions. PLATINUM partners have also used identified tactics, albeit sometimes implicitly, to (re)design student tasks and activities that foster conceptual understanding of students and promote an inquiring atmosphere. Some examples of tasks developed and used are shown below; more examples will be discussed in Section 6.4.

But before going to examples we draw attention to two other frameworks that may help lecturers create tasks that foster and assess aspects of mathematical thinking. Swan (2008) lists the following five task types that encourage concept development at secondary school level: (1) classifying mathematical objects, (2) interpreting multiple representations, (3) evaluating mathematical statements, (4) creating problems, and (5) analysing reasoning and solutions. These task types encourage students to use their innate powers of organising, classifying, characterising, examining, comparing, verifying, interpreting, evaluating, creating, expressing, analysing, and reflecting. There is no reason to believe that these task types would not serve the same purpose at undergraduate level. In addition, Swan (2008) lists the following design principles to make teaching for conceptual understanding more effective in a classroom setting:

- use rich, collaborative tasks;
- develop mathematical language through communicative activities;
- build on the knowledge learners already have;
- confront difficulties rather than seek to avoid or pre-empt them;
- expose and discuss common misconceptions and other surprising phenomena;
- use higher-order questions;
- make appropriate use of whole class interactive teaching, individual work and cooperative small group work;
- encourage reasoning rather than ‘answer getting;’
- create connections between topics both within and beyond mathematics;
- recognise both what has been learned and also how it has been learned.

Many of these principles seem valuable in a university setting as well (cf., Breen & O’Shea, 2018), but some of them may be difficult to realise in lectures to large groups of students. However, the principles seem applicable for tutorials with smaller groups of student and for the design of homework tasks.

Based on an analysis of what undergraduate students are in reality asked to do in course work and examination questions, Pointon and Sangwin (2003) identify eight classes of mathematical questions and tasks, listed in Table 6.2. In the four classes on the left-hand side, students are asked to apply knowledge in bounded situations. The

classes on the right-hand side require higher-order mathematical thinking skills. The authors notice that the latter tasks are hardly asked in reality. In PLATINUM, we regret this because these types of tasks probably promote conceptual understanding of mathematics more than the other ones. Adding more questions of this type is expected to improve the balance between procedural and conceptional learning of mathematics. This also positions the frameworks discussed in this section: whereas Pointon and Sangwin (2003) categorise questions and tasks that are actually used in school and university practice, Swan (2008) and Mason (2002) describe processes and actions to which students should be encouraged in their opinion. There are also similarities between the frameworks: ‘construct example/instance’ in Pointon and Sangwin’s taxonomy is more or less the same as ‘exemplifying’ and ‘specialising’ in Mason’s framework.

1. Factual recall	5. Prove, show, justify (general argument)
2. Carry out a routine calculation or algorithm	6. Extend a concept
3. Classify some mathematical object	7. Construct example/instance
4. Interpret situation or answer	8. Criticize a fallacy

TABLE 6.2. Pointon and Sangwin (2003) task classification scheme.

Let us continue now with some tasks and activities that have been developed by PLATINUM partners to foster conceptual understanding of students and promote an inquiring atmosphere. These examples have also been used at PLATINUM project meetings to discuss what inquiry-based mathematics education could mean and how student inquiry could be promoted by mathematical tasks.

The first two problems (Figure 6.1) come from partners at the University of Agder (UiA) and is about the use of nonstandard problems in an ordinary differential equations course (see also Rogovchenko et al., 2018, and Chapter 11). These are unusual problems for which “students have no algorithm, well-rehearsed procedure, or previously demonstrated process to follow.”

Sample problem 1

- a) Verify that $y(x) = \frac{2}{x} + \frac{C_1}{x^2}$ is the general solution of a differential equation

$$x^2 y' + 2xy = 0$$

- b) Show that both initial equations $y(1) = 1$ and $y(-1) = -3$ result in an identical particular solutions. Does this fact violate the Existence and Uniqueness Theorem? Explain your answer.

Sample problem 2

- a) Verify that $y(x) = C_1 + C_2 x^2$ is the general solution of a differential equation

$$x y'' - y' = 0$$

- b) Explain why there exists no particular solution of the above equation satisfying initial conditions $y(0) = 0$; $y'(0) = 1$.
- c) Suggest different initial conditions for this differential equation so that there will exist exactly one particular solution of a new initial value problem. Motivate your choice.

FIGURE 6.1. UiA examples of nonstandard ODE tasks.

The following words from Mason’s framework (2002) can be recognised: verify (1a, 1b, 2a), explain (1b, 2b), and specialise (verbalised in problem 2b as “Does this fact violate ...” and in problem 1c as “Suggest different initial conditions such that ...”). In terms of Swan’s framework (2008), students are in these two problems mainly invited to evaluate mathematical statements. In terms of the taxonomy of Pointon and Sangwin (2003), students are asked to interpret a situation or answer (1a, 2a, 2b), to show/justify (1b), and to construct an example/instance (2c).

Partners from the University of Amsterdam (UvA) have used the tactic of turning an existing textbook question into a more inquiry-based question (cf., Dorée, 2017) for several problems in an analysis course for first-year mathematics students. Here we only discuss the following original problem (Ross, 2013, Exercise 14.7):

Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

Past experience of tutorial lecturers is that this is a fairly difficult exercise for students unfamiliar with the subject: you need to treat small and large values of n separately. Many students do not get this idea and are already lost at the start of the proof. In order to guide students, the new exercise (Figure 6.2) starts with the special case $p = 2$ and students are asked to consider the magnitude of the squares compared to the original sequence. Once they understand this case, they can use it to prove the specialised statement, and hereafter generalise towards arbitrary p , including the case $0 < p < 1$.

Sample problem 3

Let $\sum a_n$ is a convergent series of nonnegative numbers.

- a) For how many values of n can we have $a_n^2 > a_n$?
- b) Show that $\sum a_n^2$ converges as well.
- c) What can you say about a_n^p for $p \in (0, \infty)$?

FIGURE 6.2. UvA example from an elementary analysis course.

In terms of Mason’s framework (2002), the task designers first specialise the original statement to the case $p = 2$ in the hope and expectation that students can hereafter see the general approach to proof from the particular case (specialising to help generalising). Instead of asking to prove a theorem, they ask students in the third subtask to make a conjecture for the general case with $p \in (0, \infty)$. Of course students must justify their statement. In terms of Swan’s framework (2008), students are invited in the revised exercise to evaluate mathematical statements and to analyse reasoning and solutions (actually analysing their own reasoning in the special case). In terms of the taxonomy of Pointon and Sangwin (2003), students interpret a situation or answers (3a) in the special case $p = 2$ and prove the statement in this special situation (3b) before they extend this to the general case (3c).

The last two examples of inquiry-based tasks (Figure 6.3), which illustrate the use of models of processes and actions, are taken from instructional materials of partners at Loughborough University (LU) for first-year materials engineering students. The first subtask of Problem 4 is designed for use in a lecture, but all other subtasks are considered more appropriate for tutorials, preferably in the form of small group work. The computer environment GEOGEBRA³ allows students to explore mathematical situations, in particular to explore functions using multiple representations.

Because many competencies are addressed in Problem 4 it comes to no surprise that, in terms of Mason’s framework (2002), many powers of students are triggered in

³www.geogebra.org

Sample problem 4

- a) Consider the function $f(x) = x^2 + 2x$ (x is real). Give an equation of a line that intersects the graph of this function (i) Twice (ii) Once (iii) Never.
- b) If we have the function $f(x) = ax^2 + bx + c$, what can you say about lines which intersect this function twice?
- c) Write down equations for three straight lines and draw them in GEOGEBRA. Find a (quadratic) function such that the graph of the function cuts one of your lines *twice*, one of them *only once*, and the third *not at all* and show the result in GEOGEBRA.
- d) Repeat for three different lines (what does it mean to be different?)

Sample problem 5

Use sliders in GEOGEBRA to determine which of the graphs on the right could represent the function

$$y = ax^4 + bx^3 + cx^2 + dx + e$$

Here a, b, c, d and e are real numbers, and $a \neq 0$. Explain your thinking.

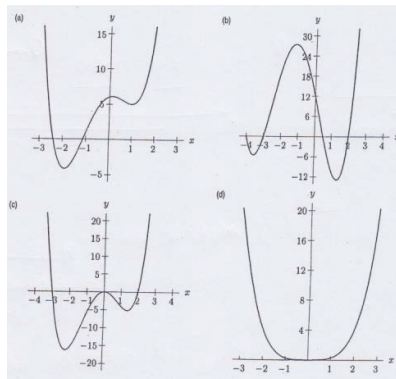


FIGURE 6.3. LU examples of inquiry-based tasks using GEOGEBRA.

these subtasks: exemplifying, specialising, generalising, comparing, organising, varying, conjecturing, explaining, justifying, verifying, imagining, and depicting. This is typical for an inquiry-based task. Mason and Johnston-Wilder (2006) actually recommend the use of a ‘mixed economy’ of tasks in order to realise as many goals as possible because no single strategy or task type has proved to be universally successful in developing mathematical thinking. In terms of Swan’s framework (2008), students are invited to classify mathematical objects (4b), to interpret multiple representations (4a, 4c, 4d), and to evaluate (their own) mathematical statements (4b). In terms of the taxonomy of Pointon and Sangwin (2003), students classify some mathematical object(4b), interpret a situation or answer (4a), show and justify outcomes(4c, 4d), and construct instances/examples (4a,4c,4d).

6.3. Documentation of Inquiry Tasks in PLATINUM

Problem 5, taken from (Hughes-Hallett et al., 2005, p. 47), is a guided inquiry task designed with the intention that students use GEOGEBRA to experiment with coefficients in equations and scales on axes to gain insights into mathematical relationships and that lecturers/teaching assistants circulate among groups observing activity, encouraging work on tasks, probing students’ mathematical thinking, and discussing students’ ideas. In Mason’s framework (2002), the tactic ‘say what you see’ is expected to help students make progress while they are (hopefully intentionally) manipulating sliders in order to get a sense of what is going on then in terms of the graphic representation of the polynomial and over time be able to articulate this sense in a mathematical way. This task is not meant to be a random exploration because students are asked to explain their thinking during the classification process of which graph can be constructed from a fourth degree polynomial function. Explanation

works best, especially for the person who explains, if there is someone else to explain to. This is why this task is actually meant for group work, even though the task itself does not demand it, and that students give feedback to their peers, lecturers and/or teaching assistants. This makes a student task distinct from a student activity: like Mason and Johnston-Wilder (2006) we consider in this chapter a task as being what students are asked to do, whereas an activity means what students actually do in their interaction with peers, lecturers, resources, environment, and so on around the task.

A mathematical task initiates mathematical activity of a student: it sets the direction of the student thinking and acting, influences the level of student engagement, and determines to a large extent what a student learns. However, a task is actually no more than a means to steer a student toward meaningful learning and practising of mathematics. There is no guarantee that a student will work as planned by the task designer and achieve the intended learning outcomes. Mason (2002, p.105) uses the following words to express the importance of careful task design and that the task itself does not automatically lead to the intended mathematical activity of students and/or the realisation of the set pedagogic purpose:

In a sense, all teaching comes down to constructing tasks for students, because most students believe (however implicitly) that their job as a student is to complete the tasks they are set, including attending sessions and sitting examinations. This puts a considerable burden on the lecturer to construct tasks from which students actually learn.

Rephrasing Watson et al. (2013, p. 10), tasks generate student activity which affords opportunity to encounter mathematical concepts, ideas, strategies, methods and techniques, to use and develop mathematical thinking and modes of inquiry, and to form a view of mathematics. In the PLATINUM project we are in particular interested in the design and use of tasks and activities that promote conceptual understanding of mathematics through student inquiry. The objective of Intellectual Output 3 in this Erasmus+ project is to

- develop a collection of teaching units that promote mathematics conceptual learning through an inquiry approach;
- synthesise working models from the designs of teaching units;
- use teaching units in specific regular courses and to collect data about their use;
- explore possibilities to make teaching units accessible for students with identified needs; and
- package and present teaching units for a wide international audience of teachers, teacher trainers, and educators with an interest in IBME.

What is the meaning of *teaching unit* within the PLATINUM project? First we note that student inquiry is not necessarily restricted to a single event with a single task, perhaps divided in subtasks. Just like subtasks in a single task, the earlier tasks in a task sequence are meant to provide students experiences that scaffold them in the solution of later tasks, allowing them to engage in more sophisticated mathematics that would otherwise not have been possible. This is certainly important for students who are not yet well trained in mathematical fundamentals, still need to learn mathematical concepts relevant for a student inquiry, and can benefit from support of lecturers and task designers to establish this mathematical grounding. Being aware that bachelor students most likely do not have the mathematical experience to ask the questions and follow the directions that lecturers of mathematics spontaneously engage with, PLATINUM partners have been trying to stimulate inquiry for students while they learn the basics of mathematics in calculus, linear algebra, and so on. This cannot

be achieved in a single task, but requires a teaching and learning path with multiple tasks. Herein not only the task sequence matters, but also the intended mathematical activity, and the pedagogic purpose. In a PLATINUM teaching unit these three aspects come together and are documented to inform others interested in the student inquiry or lecturers who use the teaching unit, learn from this use, and try to improve it. For the task design phase this resembles the notion of a hypothetical learning trajectory introduced by Simon (1995), which is “made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning processes—a prediction of how the students’ thinking and understanding will evolve in the context of the learning activities” (p. 136). In developmental research, a hypothetical learning trajectory is cyclically adapted and improved on the basis of experiences with the trajectory in teaching practice. This is the approach that most communities of inquiry at the PLATINUM partner universities have chosen and that they describe in more detail in their case studies in Part 3 of this book.

The work done by lecturers in the PLATINUM project also illustrates the important role they play in the design of teaching units as inquirers who explore

- the kinds of tasks that engage students and promote mathematical inquiry;
- ways of organising the learning situation that enable inquiry activity; and
- the many issues and tensions that arise related to the discipline, classroom, colleagues, and educational system;

and who reflect on what occurs in practice with feedback to future action. An inquiry cycle of teaching adopted from (Jaworski, 2015) in PLATINUM to characterise the work of lecturers-as-inquirers in the design of teaching units is shown in Figure 6.4.

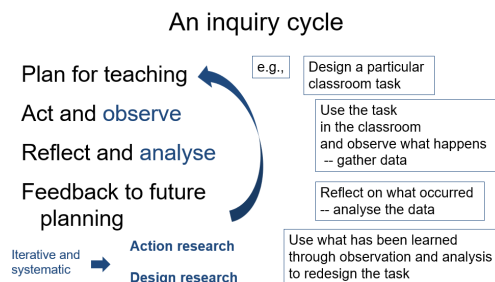


FIGURE 6.4. An inquiry cycle used in PLATINUM for the design of teaching units.

The three-layer model of inquiry outlined in Chapter 2 (see also Jaworski, 2019) distinguishes the following three forms of inquiry practice that involve students, lecturers and educators:

- *Inquiry in mathematics*: university students learning mathematics through exploration in tasks and problems in classrooms, lectures and tutorials; lecturers using inquiry as a tool to promote student learning of mathematics;
- *Inquiry in mathematics teaching*: lecturers using inquiry to explore the design and implementation of tasks, problems and activity in classrooms; educators using inquiry as a tool to help lecturers develop teaching;
- *Developmental research inquiry*: lecturers and educators researching the processes of using inquiry in mathematics and in the teaching of mathematics.

This is too much inquiry work for a single person to do professionally and too hard to maintain under pressure of other job obligations. This is why the notion of community of inquiry (CoI) has been adopted in the PLATINUM project, as discussed in

Part 1. By working together in a community, each might learn something about the world of the others, and equally important might learn something more about his or her own world. Belonging to a CoI also motivates lecturers to document their inquiry activity, and in particular the design and use of teaching units. Documentation is not only useful for the lecturer, but also for the communities of inquiry to which s/he belongs. No demands or constraints have been set within the PLATINUM project for the documentation of teaching units. Communities of inquiry were only given a template with aspects to which they could pay attention in the documentation and some concrete examples of documentation written at an early stage by the UvA CoI. The main reason for this freedom in documenting the teaching units, but still providing a template for guidance has been that many similarities and differences were identified amongst the communities of inquiry concerning ambitions/scope, study programmes and target groups, mathematical concepts/contents, envisioned use of digital technology, and planning of tasks and timeline. Shared interests and goals of partners were in improving the learning of mathematics in relevant contexts, increasing authenticity in student activities (includes use of digital technology), improving students' understanding of mathematical concepts, methods and techniques and their roles in applications, introducing inquiry-based activities in mathematics courses, and in innovating instruction (e.g., to increase student motivation and engagement). Some partners were thinking of modifying existing courses (UiA, multivariable calculus; LUH, discrete mathematics) or starting from scratch new courses (UvA, basic mathematics for biomedical sciences, analysis of neural signals), while others were planning to modify units/topics within existing courses to make them more inquiry-based (e.g., BGKU, sequences and series; BUT, complex functions; LU, complex numbers; MU, optimisation; UCM, special forms of matrices). Most plans were made for bachelor study programmes with quite often large numbers of students participating in the pedagogic cases (typical for programmes in engineering, economy, and life sciences), but there were also plans presented for courses with a small number of master students (e.g., in pre-service teaching training). It turned out that there was a large variety in mathematical concepts treated in the pedagogic case ranging from complex functions (BUT), complex numbers (LU), basics of discrete mathematics (LUH), differential equations (MU, UCM, UiA, UvA), logic (UCM), mathematical modelling (BGKU, UiA, BUT), matrix theory (UvA, UCM), multivariable calculus (UiA, UvA), sequence, series and limit (BGKU, UvA) to statistics/regression (UCM, BGKU). Besides virtual learning environments (MOODLE, CANVAS, ...) and smartboards, beamers, voting systems, and so on, many different mathematical software environments (mostly mainstream software for higher education) were envisioned to be used by students in their work with the developed teaching units, ranging from AUTOGRAPH⁴ (LU), EXCEL (BGKU), GEOGEBRA⁵ (LU, MU, UvA), MAPLE (UCM, UiA, BUT), MATHCAD (BGKU), MATHEMATICA/WOLFRAM ALPHA (BUT, BGKU), MATLAB⁶ (UCM, UvA), MAUDE⁷ (UCM), RSTUDIO⁸ (UCM, UvA) to SOWISO⁹ (UvA). We actually consider the variety of pedagogic cases as a strong point of the PLATINUM project because in this way inquiry-based mathematics education could be explored in various university teaching practices.

⁴<https://completemaths.com/autograph>

⁵www.geogebra.org

⁶www.mathworks.com

⁷doi.org/10.1016/j.jlamp.2019.100497

⁸www.rstudio.com

⁹www.sowiso.com

Each PLATINUM teaching unit has been built around one mathematical topic, is designed for student inquiry, and is used in higher education classroom practice. These teaching units serve as exemplary materials for mathematics lecturers and for instructors in professionalisation programmes to experience inquiry-based mathematics education (IBME) at university level and to inspire further development of IBME. The documentation of each teaching unit consists of (1) information for lecturers, (2) information about the learning activities, and (3) the worksheets and files used in the classroom, plus supplementary material. Some items in the information for lecturers are:

- *Unit description*: a short description of the unit about its subject matter and organisation, the student level, expected prior knowledge, the significant mathematical concepts and essential questions addressed, the course and context in which it has been used in HE practice, and the estimated duration;
- *IBME character of the teaching unit*: the kind of student inquiry that is applied and the addressed inquiry abilities;
- *Technological Pedagogical Content Knowledge (TPACK)*: the common students' difficulties and alternative conceptions that have been identified by mathematics education research and/or by lecturers in higher educational practice, and the role of ICT in the teaching unit;
- *Lecturers' experiences in the teaching practice*: a short reflection about its use within HE classroom practice (which expectations were met or not, challenges encountered in the implementation, students' reactions, ...).

Information about the learning activities in a teaching unit consists of short descriptions of learning objectives, main concepts and essential questions, envisioned student engagement in the construction of conceptual understanding, and of tool use for each learning activity. The third part of the documentation of a teaching unit consists of

- *student tasks and worksheets*, in source format (WORD, L^AT_EX, ...) and in PDF format;
- *auxiliary files* such as data files, software-specific files, simulation files, assessment sheets, reference materials, and so on; and
- *supplementary files*, for example, more detailed notes about the design of the unit and the activities, classroom experiences, related narratives, etc.

The template for documenting a teaching unit for student inquiry and all documented PLATINUM teaching units can be found in the website of this Erasmus+ project.

6.4. Examples of Inquiry Tasks Developed and Used in PLATINUM

In this section we present in detail three examples of inquiry-based tasks developed and used in PLATINUM. They are selected to represent typical designs and approaches of student inquiry that can be used when teaching mathematics to first-year undergraduates.

6.4.1. Exploring Data-Driven Numerical Differentiation. This example is taken from the Basic Mathematics Module for Biomedical Sciences developed by the UvA partners. The entire module, discussed in more detail in Chapter 12, can be seen as a learning trajectory to introduce Systems Biology to first-year students of biomedical sciences. In Systems Biology, biological processes of change are modelled by differential equations and values of parameters in these models are estimated by comparing modelling results with measured data. But in order to be able to do this estimation one must be able to compute values of derivatives of the modelled quantity. Students investigate early in module how to compute the numerical data. First they

are challenged in a lecture to form small groups of two or three students and come up themselves with ideas how to do this (they had to suggest at least two possibilities). The task shown in Figure 6.5 is used for that purpose.

Given are the following values of a function $y(t)$ in the neighbourhood of $t = 1$:

t	0.7	0.8	1.0	1.1	1.2
$y(t)$	0.741	0.819	1.000	1.105	1.221

What is the best approximation of $y'(1)$?

(exact answer = 1 because the used function is $y(t) = e^{t-1}$)

Try several methods and compare the results with each other.

FIGURE 6.5. An inquiry task used in a lecture.

The lecture part of the teaching unit, with the invitation to compute a derivative at a point on the basis of few surrounding data points, can be characterised as guided inquiry, meaning that there is no predetermined method, but that students must determine how to investigate the problem and find answers to the question raised by the lecturer. By raising the question in a classroom discussion, the students are expected to be intrigued and tuned in on the exploration of mathematical methods. Preferably, they do not do this individually but with peers. The goal is that students experience that by talking about mathematics with each other, their own thinking becomes deeper and fruitful.

Numerical differentiation is a subject that is suitable for a more open inquiry approach when students are familiar with the concepts of a derivative at a point, tangent line, and difference quotient as approximation of a derivative at a point in the domain of some mathematical function. One might expect that they can then indeed come up with the forward finite difference as a numerical approximation of a derivative at a point. This seems a good starting point to let students discover other ways to numerically approximate a slope at some point. Students are invited to discuss for about 20 minutes possible approaches with peers in small groups. Methods and results students come up with are then discussed in classroom: it is expected that they can propose a backward finite difference method and a combination of the forward and backward difference method. The discussion offers the opportunity to pay attention to what underpins mathematical methods and why it is common in mathematics to look for alternative methods and techniques for solving the same problem and to explore what works best and under what conditions. It is important that there are many possible methods because inquiry means asking questions and seeking answers, raising follow-up questions and seeking more answers, recognising possibilities, explore options, discuss pros and cons, and so on. There should not be an early end point in student inquiry and in the discussion about mathematical methods.

After the lecture, students implement their methods in RSTUDIO during a practice session in order to further explore the numerical methods regarding accuracy, efficiency, coping with noise in real data, and so on. The tutorial in which students implement standard finite difference methods for numerical differentiation and explore the advantages and limitations of the methods is an example of structured inquiry, meaning that students follow more or less directions to implement 2-point and 3-point difference methods and set up a numerical experiment to explore by example which method gives better results with data that are noisy. In Table 6.3 we typify these student activities in terms of the 7E learning cycle of student inquiry.

<i>Assignment</i>	<i>Activity</i>	<i>E-Emphasis</i>
1.	Plotting a function	Elicit
2.	Plotting a function and its derivative in one diagram	Elicit
3.	Implementing the 2-points and 3-points numerical derivative	Engage
4.	Exploring the effect of step size and data noise on numerical differentiation	Explore

TABLE 6.3. Characterisation of student activities in the numerical differentiation practice session in the teaching unit of UvA partners via the 7E-instruction model of Eisenkraft (2003).

At completion of the teaching unit, students are expected to have strengthened their abilities to

- talk about and work with the concept of function when it is merely presented in the form of function values;
- understand why one would be interested in a numerical derivative;
- compute numerically the rate of change of a quantity when only data are given instead of a formula;
- carry out computations of numerical derivatives in RSTUDIO;
- develop investigations (numerical experiments) in order to inspect and explain the accuracy and efficiency of numerical differentiation methods; and
- think more critically about mathematical methods and techniques.

These abilities contribute to what Goodchild et al. (2021) call a ‘critical stance’ toward learning and teaching of mathematics, which is complementary to critical alignment. The notion of critical stance is according to these authors distilled into three components: awareness, self-evaluation, and agency:

Stance, we assert, is a mode of ‘being’ an attitude, perspective or disposition. Critical stance is dependent upon the student’s awareness, the information and experience they possess to reach an informed judgment about an issue, and recognition of their agency to make a difference. Critical alignment to a practice relates to a person’s relationship with the practice. On the other hand, critical stance also relates to the personal characteristics and attributes that the person brings to their participation.

In the student activities described in this example the designers try to give students opportunities for critical awareness and reflection on one’s own experience, meanings, and knowing. By letting students come up themselves with various methods for computing a numerical derivative and explore the effectiveness of various methods they can recognise that one method is from mathematical point of view more sophisticated and effective than another, and that one can be on the one hand critical about mathematical methods but on the other hand have agency to change or try-out things in investigations on the basis of own reflection and evaluation of experiences.

6.4.2. Exploring Properties and Rules of Probability. The following example is again a small teaching unit for use in a lecture and aimed at steering students away from passive listening to the lecturer toward active learning via hands-on/brains-on activities. It comes from partners at Masaryk University (MU), who developed it for a statistics course in the first-year study programme of Business and Economics.

Students work for about half an hour in small groups during a lecture. They use an A4 sheet with all possible outcomes of a roll with two dice (actually four copies of Figure 6.6 are used in the worksheet shown in Figure 6.7) to carry out short inquiry tasks and they formulate their findings and conclusions.

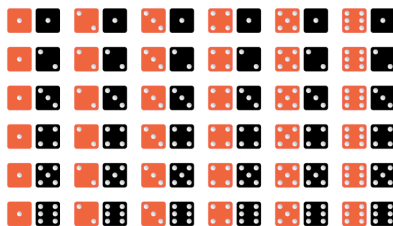


FIGURE 6.6. A dice sheet showing all outcomes of a roll with two dice. The sheet is used by MU partners in a teaching unit about properties and rules of probability.

Based on the first task sequence with the solution sheet to the right, try to replace the question mark symbol in the following relationships.

- (i) $P(A) + P(A') = ?$
- (ii) If $A_1 \subset A_2$, then $P(A_1) ? P(A_2)$
Take inspiration, for example, from events A and D.
- (iii) If $A_1 \cap A_2 = \emptyset$, then $P(A_1 \cup A_2) = ?$
- (iv) If $A_1 \cap A_2 \neq \emptyset$, then $P(A_1 \cup A_2) = ?$ Take inspiration, for example, from events A and C.
- (v) $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - ? - ? - ? + ?$
Take inspiration, for example, from events C, E and H.

*A: suma je deset
C: na jednom kostičku je jednička
E: na obou kostičkách je jednička
D: dvoje, šestka?*

$m(A) = 9$
 $m(E) = 11$
 $m(A \cap E) = 0$

$m(A) = 9$
 $m(C) = 6$
 $m(A \cap C) = 1$

$m(H) = 6$
 $m(C) = 6$
 $m(E) = 11$
 $m(H \cup C \cup E) = 2$
 $m(H \cap C) = 5$
 $m(C \cap E) = 1$
 $m(H \cap E) = 1$

FIGURE 6.7. Task sequences for students to conjecture rules of probability on the basis of results of a sample problem situation.

The tasks introduce properties and rules of probability. But instead of stating the rules and using them in an application, the MU partners chose to have a set of introductory tasks that help students conjecture rules of probability. Although these conjectures are made on the basis of one concrete situation, the drawing of two dice, it is hoped and expected that students start to understand that such examples are common in mathematical investigations to understand problem situations and come up with solutions that work in other situations as well. According to Mason’s framework (2002) this process means that specialisation is often needed to make generalisation possible.

The student activity consists of two parts: firstly, students determine the sample space of all possible outcomes for the following events in rolling two dice.

- (A) The sum of dots in a roll equals 10;
- (B) The sum of dots in a roll differs from 10;
- (C) Each dice rolled has the same number of dots;
- (D) Each dice rolled is 5;
- (E) At least the roll of one of the dice is 1;
- (F) The sum of the dots in a roll equals 10 or at least one dice rolled is 1;

- (G) The sum of the dots in a roll equals 10 or each dice rolled has the same number of dots;
- (H) The sum of the dots in a roll is less than 5;
- (I) The sum of the dots in a roll is less than 5, or each dice rolled has the same number of dots, or at least one dice rolled is 1.

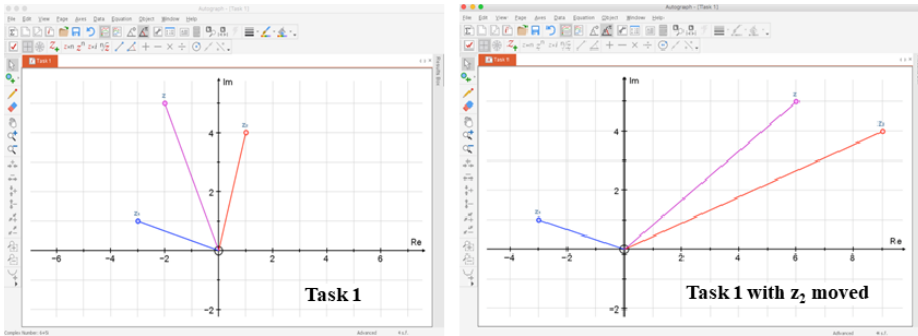
Hereafter students get the task sequence shown in Figure 6.7, in which they must conjecture basic probability formulas and underpin their conjectures.

This student work is finished with a whole classroom discussion of the proposed conjectures. The social aspect of learning and doing mathematics is considered important for students to adjust their view on mathematical inquiry.

6.4.3. Complex Number Arithmetic. The following example is part of a teaching unit for small-group work on complex numbers, which comes from a mathematics module in the Foundation Studies programme developed by partners at Loughborough University (LU) and is described in more detail in the case study of Chapter 15. Whereas traditional instruction often starts with specifying the calculation rules of complex numbers and illustrates this with examples using algebraic representations, the designers of this task have chosen to apply *reverse-engineering* of such questions and use the mathematics software tool AUTOGRAPH¹⁰ for helping students in tutorial sessions to explore complex number arithmetic in a geometric perspective and connect geometric insights with algebraic manipulation. The whole teaching unit, created together with student partners (Treffert-Thomas et al., 2019), consists of 6 tasks: (1) addition, (2) subtraction, and (4) multiplication of complex numbers, (4) complex conjugate of a complex number, and (5) squaring and (6) cubing a complex number. Here we use the original Task 1, shown in Figure 6.8, to exemplify the more general ideas of the task designers. The adaptation of this task to make it more suitable for students with identified needs will be discussed in Section 6.7.

In this task students see three complex numbers on the computer screen, labelled z_1 , z_2 and z , and one of the complex numbers (z_1) is specified in the question text. They must figure out what happens when they move z_2 and in this way try to give a geometric interpretation of the relationships between the shown complex numbers. No reference is made here to calculations or algebraic manipulations. AUTOGRAPH is used as a tool to visualise the mathematical relationship, but it is left up to students to make the link. A *reverse engineering* approach is used in this task, meaning that instead of asking the straightforward question “What is the sum of z_1 and z_2 ?” with only a correct or wrong answer and no scope for investigation, students are asked to move z_2 to the position so that the sum with z_1 reaches a particular position in the complex plane. Only in a later subtask (c) are students invited to undertake some associated calculation by hand in the hope and expectation that they relate movements on the computer screen to the written work and the theory involved. In subtasks (d) to (f), students are explicitly invited to reflect on specific results to develop more general awareness of complex number concepts related to addition. In terms of Mason’s framework (2002), students are asked in Task 1 to apply the tactic *say what you see* in a special case, to explore more special cases to see a pattern, and then to generalise their findings. For the explorative phase, no suggestions are made in the tasks; students work independently and follow their own strategy. Tutorial lecturers circulate in the classroom, listen to what goes on in group work, encourage students, and lead whole-classroom discussions. This collaborative aspect is part of the pedagogic use of the task and not explicitly stated in the task itself.

¹⁰<https://completemaths.com.autograph>



Task 1

There are three complex numbers labelled z_1 , z_2 and z .

z_1 is to be kept fixed while z_2 and z can be moved.

Select z_2 and move it until z reaches the position $6 + 5j$.

- What complex number is z_2 ? Right click and “Unhide All” to check your answer. The correct answer appears in green.
- What is the relationship between z_1 , z_2 and z ?
- Now calculate by hand:
With $z_1 = -3 + j$ and $z = 6 + 5j$, find z_2 such that $z_1 + z_2 = z$.
- Re-load *Task 1*. Move z_2 around the screen and notice how z changes as a consequence. What is the geometric connection between z_2 , z and the complex number z_1 (which has stayed the same during your movements)?
- Now you are allowed to move both z_1 and z_2 . Move these to different locations but make sure that z still ends up being $6 + 5j$. Make note of the positions of z_1 and z_2 . Does your geometric connect from (d) still hold?
- Repeat another four times so that you have five different pairs of values for z_1 and z_2 with each of them making z to be at $6 + 5j$. For all of these, what is the relationship between z_1 , z_2 and z and does your geometric relationship still hold for each of them?

FIGURE 6.8. Screen shot of AUTOGRAPH files and instructions for the original Task 1 in the complex number arithmetic teaching unit, used in the Loughborough Foundation programme.

6.5. Use of ICT in Student Inquiry

Much research has been done about the use of ICT in mathematics education, especially at primary and secondary school level, and it has offered a range of theoretical perspectives. Two volumes of the National Council of Teachers of Mathematics (Blume & Heid, 2008; Heid & Blume, 2008), the 17th ICMI study (Hoyles & Lagrange, 2010), books in the Springer series called ‘Mathematics Education in the Digital Era’ (e.g., Leung & Baccaglini-Frank, 2017), and the proceedings of the International Conference on Technology in Mathematics Teaching (ICTMT) are good sources of information. Many lessons have been learned; the most important ones are that

- use of ICT for improvement of the depth and quality of mathematics learning is much more complicated than initially anticipated by proponents of tool use;
- ICT tools serve at a more fine-grained level many different goals in teaching and learning of mathematics;
- task design of ICT-enhanced mathematical activities is a delicate, multi-faceted issue; and
- the terrain of technology-supported education is rapidly changing and offering new ways of engaging with mathematical thinking, but with didactic theory development hardly keeping up with technological progress.

ICT use in inquiry activities for students is even more complex because of the twofold nature of inquiry learning, which can be described as *inquiry as ends* and *inquiry as means*. The first of these sees inquiry as a set of instructional outcomes for students that involve understanding of inquiry and abilities to do inquiry. In the perspective that university study programmes should enable their students to become literate in mathematics and ICT at the level that their discipline requires, the dominant idea is that students should learn to use ICT tools that are commonly used in their profession for doing mathematics. The use of R and RSTUDIO in the basic mathematics and statistics course for biomedical sciences students, described in Chapter 12, is a typical example in which this perspective plays an important role.

The second aspect of inquiry, inquiry as means, is related to inquiry as an instructional approach or pedagogy. The PLATINUM objective to promote conceptual understanding through student inquiry is an example of this perspective. Teaching units designed for this purpose use ICT tools as means to realise instructional goals as best as possible. Task design emphasises in this case the mediating role of the tools. In this section we look in detail at a PLATINUM example of this type of use of ICT, namely the teaching unit about isometries and tessellations of the Euclidean plane which has been developed for first-year mathematics programmes by the UCM partners (Sáiz, 2020) and uses the dynamic mathematics environment GEOGEBRA. But before doing this, we would like to stress that, despite the apparent distinction between tool use in inquiry learning at university level, the two modes of tool use are better not treated as opposite modes because one cannot do without the other: without mathematical knowledge and skills and without inquiry abilities students will not learn much from ICT-enhanced inquiry and, conversely, a scientific context is always needed as a practice arena for inquiry abilities. For example, in Chapter 12, UvA partners describe how the use of R and RSTUDIO enables their students to learn basic concepts of Systems Biology in ways that would otherwise not be possible.

In this section we adopt the model of Pedaste et al. (2015) for IBME activities, consisting of the phases Orientation, Conceptualisation, Investigation, Conclusion, and Discussion, to discuss the use of ICT in student inquiry in these phases and in particular in the teaching unit about isometries and tessellations of the Euclidean plane developed by UCM partner as this may serve as a prototypical example. This teaching unit, which takes about 5 hours of student work,¹¹ consists of two parts: thinking and learning about (1) planar isometries and (2) crystallographic groups and tessellations in the plane. For details we refer to the documentation of this teaching unit, which is available in the PLATINUM website (<https://platinum.uia.no>).

In the orientation phase, students are introduced to a domain of knowledge or a subject of study. Tasks in this phase are designed to activate students' prior mathematical and disciplinary knowledge, raise interest in the subject (relevance to the discipline), and relate to the students' background (e.g., skills, culture, and language). Their main aims are to enable students to explore and analyse a given problem situation. ICT is in this phase typically used to practise prior skills and to provide microworlds or simulations for initial exploration of the subject. The first part of the UCM teaching unit about isometries, lasting about one hour, serves this purposes. The dynamic geometry environment GEOGEBRA is in the first activities used to visualise the effect of transformations on points and triangles so that students can draw their own conclusions. Students are not given full access to the GEOGEBRA environment, but instead get tailor-made GEOGEBRA applets to explore properties; see for

¹¹Duration of work on the teaching unit about tessellations depends on whether students also create their own tessellations and/or explore work of the Dutch artist M.C. Escher.

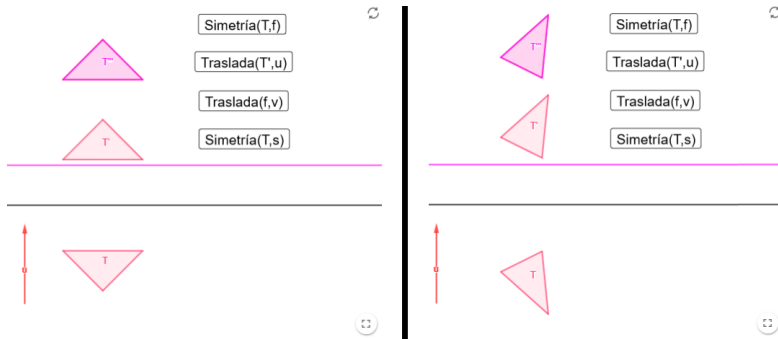


FIGURE 6.9. GEOGEBRA applet created by UCM partners for the composition of a reflection and a translation in a direction perpendicular to the mirror line. The result is a reflection in the translated mirror line. Students can press dedicated buttons to carry out transformation on objects in the plane, observe what happens, and formulate a hypothesis. On the left-hand side results of the initial settings of the applet are shown; on the right-hand side results are shown for a more general triangle obtained by dragging the original triangle.

example Figure 6.9 for two screen shots of an applet for composition of a reflection and a translation in a direction perpendicular to the mirror line. The task designers provide the students in this way with a microworld that (hopefully) helps them focus in their work on the mathematical properties instead of the technicalities of the computer environment. In terms of the framework of Kaput (1992) on computer use in education, ICT is in this case for the task designers a toolmaker/mediummaker and for the students an educational medium. The introductory activities about isometries also prepare the students for using GEOGEBRA in their prospective inquiry work in the second part of the teaching unit.

In the second part of the teaching unit, students explore planar tessellations, also known as wallpaper patterns. A wallpaper pattern is a way to cover a flat surface with a repeating pattern of shapes such that there are no overlaps or gaps and a translational symmetry in two independent directions can be identified. Its symmetries can be viewed as planar isometries and together they form a group, the symmetry group of the pattern. Seventeen symmetry groups of planar patterns can be distinguished (see, for example, Schattschneider, 1986). In the teaching unit seventeen GEOGEBRA applets have been created, one for each symmetry group, and most of these activities are inspired by the work of the Dutch artist M. C. Escher (1958)¹² The task designers connect mathematics with art in the hope and expectation that this motivates students in their inquiry and let them study the underlying mathematics in an attractive way. This is important for students as it helps them persevere as they engage in studying the wallpaper patterns.

The first three GEOGEBRA applets allow students to visualise in a detailed way how two of Escher's tilings (Seahorse, No 88; Beetle, No 91) can be created from a single tile by repeated application of generators of a matching group of isometries. Figure 6.10 shows two screen shots to construct from an initial geometric shape (a parallelogram) containing some black lines via rotations and translations a basic tile

¹²It is funny to see that Spanish task designers are inspired by a Dutch artist who himself got inspired by islamic geometrical art during his visit to the Alhambra in Granada, Spain.

(a seahorse) that can then be used to create a wallpaper pattern of seahorses with symmetry group labelled p2 in the notation of the International Tables for X-ray Crystallography.

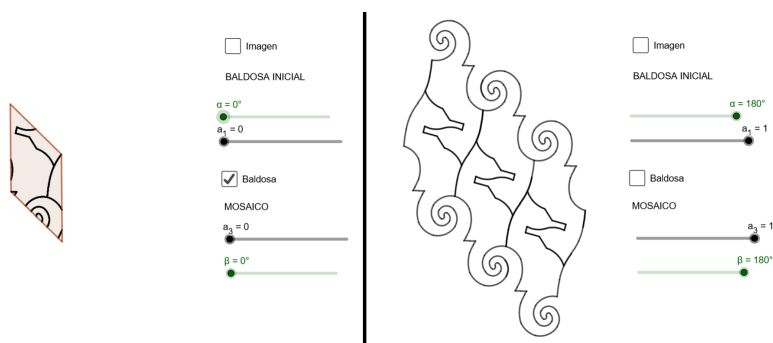


FIGURE 6.10. GEOGEBRA applet created by UCM partners to help students visualise in a detailed way how a seahorse wallpaper pattern with symmetry group labelled p2 can be constructed from an initial geometric shape containing lines. On the left-hand side is a screen shot with the initial shape and on the right-hand side is a screen shot with created seahorse shapes.

The construction of the basic tile is not explained in the applets; students have to find this out by dragging the sliders acting on the initial shape and observing what goes on. The upper slider rotates every black line that intersects the inner part of the upper edge of the parallelogram about the midpoint of the upper edge, and at the same time rotates every black line that intersects the inner parts of the left or lower edge of the parallelogram about the midpoint of the lower edge. Hereafter the lower slider acting on the initial shape translates all black lines that intersect the right edge of the parallelogram and its imaginary extension along the vector from the lower right vertex to the lower left vertex of the parallelogram, and at the same time translates all black lines that intersect the left edge of the parallelogram and its imaginary extension along the vector from the lower left vertex to the lower right vertex. The end result of this whole process, shown in Figure 6.11, is the creation of the basic tile for the wallpaper pattern, namely, the seahorse.

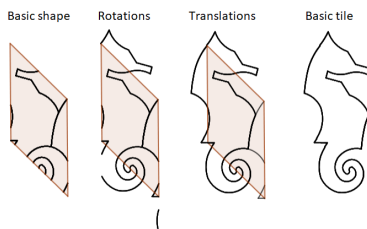


FIGURE 6.11. Screen shots that illustrate the creation of the basic tile (a seahorse) for the wallpaper pattern from an initial shape through a two-step procedure involving rotation and translation.

The inquiry is directed toward understanding the basic tile construction used by Escher in his designs (cf., Schattschneider, 2010). It is followed by the generation of parts of the wallpaper pattern by repeatedly applying generators of the matching symmetry group. In both phases of the inquiry, the dynamic nature of GEOGEBRA

helps students discover what the movement of the sliders actually means in terms of geometrical changes in the plane. Focus is in both phases on conceptualisation.

For better understanding of techniques to generate a basic tile and a wallpaper pattern from this tile students need to investigate more examples of wallpaper designs. For this purpose, the designers of the tasks have created for each remaining symmetry group a dedicated microworld that allows students to go through various stages of this process by checking options in the applet. Figure 6.12 shows the applet for investigating the symmetry group labelled $p6$, connected to Escher's tiling Flying Fish, No 99.

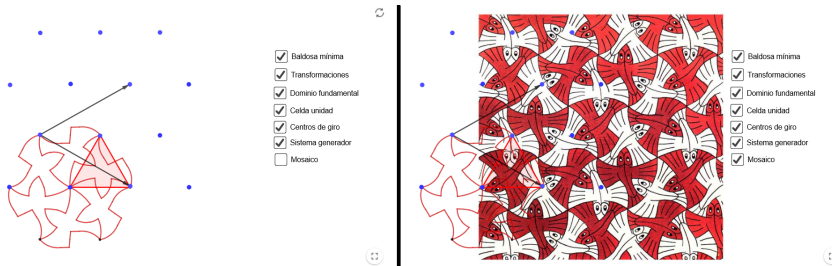


FIGURE 6.12. Screen shots illustrating the creation of the wallpaper pattern of type cmm connected to Escher's tiling Dragonfiles, No 13.

The conclusion and discussion phase of the teaching unit is a guided inquiry activity in which students can use the full toolbar to complete a wallpaper pattern of type $p6m$ with all elements for creation of this tiling (reflection axes, rotation centres, translation vectors, ...) already present in the applet; see Figure 6.13,

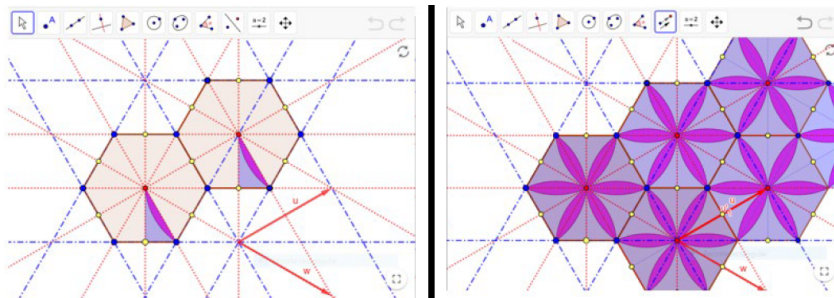


FIGURE 6.13. Screen shots illustrating the creation of the wallpaper pattern of type $p6m$ using the complete functionality of GEOGEBRA given all elements needed for the creation of the tiling.

This progressive introduction to the use of GEOGEBRA from dedicated microworld to a dynamic mathematics environment with all tools available is a deliberate choice of the task designers. They do this because in past research studies they have experienced that the wider diversity of approaches among students to explore configurations and discover new geometric properties via GEOGEBRA is accompanied by an increase in complexity of integrating technology into the classroom. Lecturer should take into account the conditions of learning mathematics with GEOGEBRA and pay attention to a genesis of and transition between figural, instrumental, and discursive reasoning (Gómez-Chacón & Kuzniak, 2015; Gómez-Chacón et al., 2016). Task designers could

help lecturers by designing effective teaching and learning paths with tasks that promote, support and sustain student inquiry. The PLATINUM project is built on the idea that the best way for lecturers and task designers to achieve these goals is to work together in a community of inquiry.

Step by step exposure of students to the full power of a dynamic mathematics environment is one of strategies that have been found effective in promoting inquiry. Another one is the inclusion of opportunities for exploration of mathematical ideas by students in learning paths, that is, by inclusion of student activities in which students pursue conceptual understanding of mathematics by posing and answering questions as they do mathematical experiments, develop strategies, make conjectures, and try to find evidence. The teaching unit of UCM partners contains plenty of such tasks and follows the strategy of gradually exploring more complex situations through GEOGEBRA applets, ending with a more open inquiry. The task sequence could still be extended with activities in which students create their own basic tiles for own designs of wallpaper patterns. This would be a fun challenge for students with artistic talents.

Dynamic mathematics environments such as GEOGEBRA have also been found effective in promoting student inquiry by dynamically linking multiple representations of mathematics objects. Part of mathematics literacy, and more generally scientific literacy, is that one has developed representational fluency. Sandoval et al. (2000, p. 6) provide the following comprehensive definition of representational fluency:

We view *representational fluency* as being able to interpret and construct various disciplinary representations, and to be able to move between representations appropriately. This includes knowing what particular representations are able to illustrate or explain, and to be able to use representations as justifications for other claims. This also includes an ability to link multiple representations in meaningful ways.

Mathematicians and scientists often use multiple representations because

- different kinds of information can be conveyed with specific types of representations (e.g., phenomena with simulations, animations, or video clips);
- interaction with multiple representations supports various ideas, strategies, and processes in problem solving;
- different representations of a problem are seldom equivalent computationally, even when they contain equivalent information; and
- use of multiple representations promotes deeper and general understanding.

We concur with Kaput (1992, pp. 533–543) that computer technology, through the dynamic linking of representations and immediate feedback, can assist students in their learning process from concrete experiences to ever more abstract objects and relationships of more advanced mathematics and science, and can support visualisation and experimentation with aspects of investigated phenomena. Ainsworth (2008) summarises a number of heuristics that could be used to guide design of effective multi-representational systems:¹³

- minimise the number of representations employed and avoid too similar representations (the coherence and redundancy principle);
- carefully assess the skills and experiences of the intended learners in order to decide on support of constraining representations to stop misinterpretation of unfamiliar representations, and to avoid unnecessary constraining representations (pre-training principle);

¹³Between brackets we place labels of the connected principle(s) of multimedia learning listed by Mayer (2020).

- select an ordering and sequencing of representations that maximises their benefits by allowing learners to gain knowledge and confidence with fewer representations before introducing more (segmenting principle);
- consider extra support like help files, instructional movies, exercises, and placement of related representations close to one another on the computer screen, to help learners overcome the cognitive tasks associated with learning with multiple representations (guided activity principle, worked-out example principle, segmenting principle, modality principles, navigation principles, spatial and temporal contiguity principle).

Several PLATINUM partners have done their best to use these design principles; you may recognise them in the described teaching unit about isometries and tessellation of the UCM partners or in the teaching unit on complex number arithmetic of the LU partners with AUTOGRAPH files that are kept as simple as possible. Figure 6.14 shows a GEOGEBRA applet used by UvA partners to illustrate how the phase plot of a parametrised differential equation depends on the value of the bifurcation parameter and what information is actually presented in the bifurcation diagram. Students (or the teacher in a lecture) drag the triangle along the axis for the bifurcation parameter, observe what happens on both sides of the applet, and draw conclusions (perhaps after first using Mason’s ‘say what you see’ tactic). The GEOGEBRA applet is designed to be as simple as possible, with no redundant information present, and with the multiple representations close to each other to make it easier to observe changes in linked representations. In other words, principles of multimedia learning are applied.

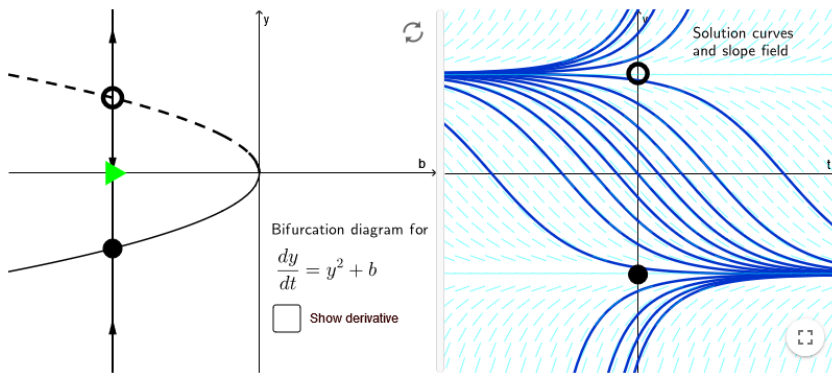


FIGURE 6.14. Screen shot of a GEOGEBRA applet used by UvA partners to connect a bifurcation diagram with changes in phase plots of a differential equation with a bifurcation parameter. Dragging the icon for the bifurcation parameter changes both sides of the applet

6.6. Guiding Design Principles Identified in PLATINUM

As we have noted before in Section 6.3 and can also be read in Chapter 2 of the book, there exist many views on inquiry-based mathematics education. Therefore it comes to no surprise that there also exist many views on what makes a good inquiry-based task for students. The examples shown in this chapter and the case studies in Part 3 of this book illustrate a great variety of inquiry-based tasks. Yet some common characteristics can be distinguished in the PLATINUM inquiry-based tasks (see also Jaworski, 2015). They

- provide easy access to mathematical ideas;
- are inclusive in the sense that they enable everyone to make a start and inspire engagement by all;
- provide opportunity to ask questions, solve problems, imagine, and explore;
- encourage discussion and reasoning;
- encourage student centrality/ownership in/of the mathematics; and
- promote mathematical thinking.

The role of lecturers and teaching assistants while a small number of students work on inquiry-based tasks can be characterised by words like

- circulating and listening;
- asking and encouraging students to ask questions;
- encouraging dialogue and/or debate;
- fostering reasoning; and
- prompting and challenging.

These ways of working are a big challenge for lecturers with large numbers of students. PLATINUM partners have in these cases often used lectures to plant seeds for student inquiry into a mathematical concepts by whole classroom discussions in which students were invited to express their ideas developed in small group work with neighbours in the lecture room. Hereafter students could dive more into the inquiry in tutorials with smaller number of students. The teaching unit about data-driven numerical differentiation presented in Section 6.5 is a good example of this approach.

The value of using ICT in mathematics education and in particular in student inquiry is manifold. Like van Joolingen and Zacharia (2009) we distinguish the following ingredients of computer-based inquiry activities:

- *a mission for inquiry* that introduces students to a domain of knowledge or subject of inquiry;
- *a source of information for inquiry* that allows students to extract relevant data needed for cognitive growth;
- *tools for expressing knowledge* in external forms;
- *cognitive and social scaffolds* to overcome the paradox that in order to learn through inquiry, one needs the abilities that are acquired through the learning itself.

In the design and implementation of ICT-enhanced inquiry activities goes much thinking and trying-out to the above ingredients.

Tasks in the orientation phase of student inquiry are designed to activate students' prior mathematical and disciplinary knowledge, to raise interest in the subject or show the relevance for the discipline, and to relate the mission for inquiry to the students' background (e.g., skills, culture, and language).

The subject of an inquiry activity is a source of information that allows students to extract relevant data. Students can obtain data from microworlds (like the GEOGEBRA applets of the UCM partners or the AUTOGRAPH files of the LU partners discussed in the previous section), data logging tools (Heck, 2012), and from modelling and simulation environments (like the use of RSTUDIO to explore dynamic systems in the UvA case study presented in Chapter 12), to mention a few. Information sources play a role in three inquiry phases of the framework of Pedaste et al. (2015), which is used by PLATINUM partners to describe modelling activities (see Chapter 8): orientation, conceptualisation, and investigation. In the orientation and conceptualisation phases data are needed to shape one's initial ideas. In the investigation phase data are needed to test and deepen ideas.

In the conceptualisation, investigation, conclusion and discussion phases of inquiry-based learning (Pedaste et al., 2015) one needs tools to provide the means for representing, processing, and analysing new data or information. How could students otherwise explain and justify their mathematical thinking, their solution strategies, and actions in open-ended activities. These tools can be mathematical representations invented or co-created by students as in inquiry-oriented mathematics education (Kuster et al., 2018) or more widely-used standard mathematical representations. They can also be mathematical constructions created in dynamic mathematics software environments like GEOGEBRA, results of computer-based modelling and simulations, a spreadsheet, report or presentation written with office tools, a computer-aided form of evidence, and so on. Thus, ICT offers opportunities for mediating the learning activities in which students engage (cf., Sfard & McClain, 2002).

The paradox that in order to learn through inquiry, one needs skills that are acquired through the learning itself, is similar to what is called the *learning paradox* (Bereiter, 1985). In ICT-enhanced teaching and learning of mathematics it means that tools enable, mediate and shape mathematical thinking, while being themselves, at least to some extent, a product of these processes. An instrumental approach to digital tool use in mathematics education (Trouche, 2020a,b) is one of the theoretical frameworks developed to address the problems that may arise when one starts to use a ready-made computer tool and explains the importance of aligning techniques that emerge in problem situation with the techniques available in the computer tool. The UvA partners have used this framework to understand the difficulties with programming in R and working with RSTUDIO of their students, and to make improvements in their instructional materials (see Chapter 12). They use cognitive scaffolds to structure R-based tasks, and they give hints and supporting information for these tasks. But such cognitive scaffolds can also be provided to students in other computer-based inquiry activities. In addition, social scaffolds can provide students with means for coordinating and streamlining collaboration with others, such as tools to visualise contributions to a shared knowledge building process, concept maps in the orientation phase of a student inquiry, a shared use of a glossary, a teacher-led classroom discussion of mathematics with a digital whiteboard for notes, figures, or mathematical representations. In case studies described in Part 3 of this book one can find accounts of classroom discussions with students during inquiry activities.

6.7. Accessibility of Teaching Units for Students With Identified Needs

One of the goals of Intellectual Output 3 of the PLATINUM project is the exploration of possibilities to make teaching units accessible for students with needs. We refer to Chapter 4 for an introduction to teaching and learning of students with identified needs. It also introduces the principles of Universal Design, a methodology adopted by PLATINUM partners to strive for an inclusive learning environment reaching the needs of as many students as possible. These principles have been worked out for an educational context as Universal Design for Learning (UDL) and general UDL guidelines are presented in Section 4.6. Below we look at how the UDL principles have guided PLATINUM partners in the design of inquiry tasks.

The first UDL principle is the use of multiple means of representation (not to be mixed up with the notion of multiple representation). Students differ in the ways that they perceive and comprehend information presented to them. At the extreme are students with impairments (e.g., those who are blind or deaf), for whom some forms of presentation are completely inaccessible. In task design one could spend time and thought on how to adapt an inquiry task for students with such identified needs. For

example, the dice sheet in the teaching unit of the MU partners cannot be used by visually strongly impaired students, but the information on the sheet could also be given in the form of a table with pairs of numbers that represent the number of dots on each dice. Such a table can be processed by a screen reader and transformed to speech output or brailled. More prevalent are students who, because of their particular profile of perceptual or cognitive strengths and deficits, find information in some formats much more accessible than others (e.g., students with dyslexia, aphasia, or mental retardation). Students coming from different cultural backgrounds and with native languages different from the instructional language used can have difficulty accessing information when words and symbols are not clearly defined. To best support all students, teaching units should include definitions of all requisite variables, symbols, and vocabulary. Certainly in the field of mathematics there is beside convention also much ambiguity in mathematical representations, and one can be best be open to students about this and emphasise that this is also a strong point of mathematical language. Anyway, the first principle reflects the fact that there is no one way of presenting information or transferring knowledge that is optimal for all students. Multiple means of representation are key. UvA partners (see Chapter 12) have for example provided several options for perception and comprehension in their instructional materials: all video clips taken from the UK Mathcentre and used in the online instructional materials offer closed captioning; GEOGEBRA applets can be reset and maximised to fill the entire screen; chapters with background knowledge such as expected prior mathematical knowledge are online available in the course material and students can practise herein skills that they were supposed to possess already; page layout includes highlighting of key words, framing of important statements and randomised examples, and hiding/opening of extra information. But in the end, multiple means of representation is not just a matter of design of instructional materials. Lecturers also play a role herein by the way they highlight critical features, emphasise big ideas, connect new information to prior knowledge, and so forth. They can lead a whole class discussion before students work through an inquiry activity to activate prior knowledge

The second useful UDL principle is the use of multiple means of action and expression. Students differ in the ways they can navigate a learning environment and express what they know. Students do not share the same capacities for action within or across domains of knowledge. Some students have specific motor disabilities (e.g., cerebral palsy) that limit the kinds of physical actions they can take, as well as the kinds of tools that they can use to respond to or construct knowledge. Other students lack the strategic and organisational abilities required to achieve long-term goals in an inquiry (e.g., students with executive function disorders or ADD/ADHD). Moreover, many students can express themselves much more skilfully in one medium than in another (using drawing tools as opposed to writing and reading print, for example). Therefore, in task design one has to make sure that there are alternatives for students' means of expression or that one maximises the accessibility of tools. For example, the UvA partners explain in their case study in Chapter 12 how they pay attention to these aspects in the design of their ICT tools. But scaffolds and supports at university level can also include optional readings, i.e., readings providing either background information or more advanced discussion of course topics, to address students with different levels of prior knowledge. Support of student planning and strategy development can be incorporated in tasks by adding questions like "Stop and think," "Make a guess," "Verify your answers," "Look for another possibility." "Give an example," and "Explain your reasoning." In terms of Mason's framework (2002) one adds questions that trigger students innate powers of mathematical thinking and doing. But

like before, multiple means of expression is not just a matter of design of instructional materials. Lecturers also play a role herein by the way they select multiple media for communication, how they guide appropriate goal-setting in an inquiry activity, how they manage information and resources, what tools they select for students to use, and so forth.

The third UDL principle is the use of multiple means of engagement. Students differ markedly in the ways in which they are engaged or motivated to learn. Some students are engaged by risk and challenge, while others seek safety and support. Some are attracted to dynamic social forms of learning and to collaboration with peers, and others shy away and prefer to work on their own. There is no single means of engaging students that will be optimal across the diversity that exists. Moreover, not all students are engaged by the same extrinsic rewards or conditions, nor do they develop intrinsic motivation along the same path. Therefore, alternative means of engagement are critical. In the design of an inquiry task, one can provide options for sustaining effort and persistence such as varying demands and resources to optimise challenge, fostering collaboration and community, clarifying expectations and structuring of group work, and increasing mastery-oriented feedback. In the design of UvA courses that use SOWISO as environment for learning, practising and assessing mathematics (Heck, 2017) increased mastery-oriented feedback is realised by providing students always randomised exercises with automated feedback. But often it is also an option to make a task more engagement-neutral. For example, in the numerical differentiation task of UvA partners shown in Figure 6.5, in the task sequence about rules of probability of MU partners shown in Figure 6.7, and in the task sequence about complex number arithmetic of LU partner shown in Figure 6.8, no words are spent on whether these are individual tasks or small group tasks. Although the task designers in PLATINUM may have thoughts about and suggestions for learning arrangements and may have specified these in documentation of the teaching unit, the decision on how to engage students is in the discussed case left to the lecturer who wants to use these tasks with her/his students.

Because more and more instructional materials become web-based and contain digital contents, task designers better look at the basic principles of web accessibility made up by the World Wide Web Consortium. This consortium organises a wide variety of recommendations for making web-content more accessible for people with disabilities (World Wide Web Consortium, 2018). Although these guidelines are made for design of web pages, they can also be generally applied to the design of any digital content (e.g., GEOGEBRA applets, simulation environments, etc.).

Multiple studies (cf., Scanlon et al., 2021, plus references herein) show that there is a world to win because many webpages used in higher education still have numerous accessibility errors and are not compliant with current Web Content Accessibility Guidelines. We expect the same for digital content in general. This is not because of unwillingness of authors to make their webpages or digital content more accessible, but is caused by lack of knowledge, unfamiliarity with principles of multimedia learning, and/or insufficient time or effort to pay enough attention to accessibility. The situation is not different for the use of Universal Design for Learning: multiple studies (cf., Schreffler et al., 2019, plus references herein) show that Universal Design for Learning is still not widely used in postsecondary STEM education after the Center for Applied Special Technology¹⁴ (CAST) introduced its first UDL Institute for educators in 1998.

We give an example from the PLATINUM project to illustrate how UDL principles can help task designers change an existing inquiry task and make it more accessible

¹⁴www.cast.org

for students with identified needs. Figure 6.15 shows a new version of the first task in the complex number arithmetic teaching unit developed by LU partners after applying some UDL principles. Looking back at the original task in Figure 6.8, there are many instructions divided over six subtasks; they are very detailed and use a lot of words. This was also reflected in the feedback from students using the task. Dyslexic students told that they lost track during their work. UDL principles and guidelines are of help here. The first thing one could do is reduce the number of instructions and length of text so that it is shorter, there is less to read, and it seems that there is less to do, even though the task overall has not changed (only the task presentation has changed).

The next improvement is the addition of approximate times, for each subtask and the task overall. This helps students with autism spectrum disorder including Asperger syndrome, who need a bit more structure and like to know some boundaries in the time to spend on tasks; otherwise they might end up spending too much time. But it also helps students in general, because managing the time in an activity is something many first-year students still have to get used to. Giving a time limit for a task one helps students better understand how far to take the task. But setting time limits for the subtasks and the task overall also helps a task designer or lecturer think about how realistic the demands on students are given the time constraints of study.

A further improvement is adding colours to the variables and mathematical formulas in the instructions and letting them match to the colours in the AUTOGRAPH files. It is often helpful for dyslexic students to keep track of their work. But the same holds for students in general: adding colouring may help reduce cognitive load while extracting information from multiple linked representations (cf., Ainsworth, 2008).

New Task 1: less wordy, with times and colours

Task 1: (Total time 15-20 mins.)

Open the AUTOGRAPH file *Task 1*.

There are three complex numbers labelled z_1 , z_2 and z .

z_1 is to be kept fixed while z_2 and z can be moved.

Select z_2 and move it until z reaches the position $6 + 5j$.

- (a) What complex number is z_2 ?
Right click and “Unhide All” to check your answer. (2-3 mins.)
- (b) What is the geometrical relationship between z_1 , z_2 and z ?
(2-3 mins.)
- (c) Now calculate by hand: With $z_1 = -3 + j$ and $z = 6 + 5j$, find z_2 such that $z_1 + z_2 = z$. (2-3 mins.)
- (d) Re-load *Task 1*. Move z_2 around the screen and notice how z changes. Describe the position of z in relation to z_1 and z_2 . (5 mins.)
- (e) Explore this relationship. Move z_1 and z_2 to different locations but make sure that z still ends up being $6 + 5j$. Does what you thought in (d) still hold? (5 mins.)

FIGURE 6.15. Instructions in Task 1 about addition of complex numbers, with the same AUTOGRAPH files as in Figure 6.8, after application of some UDL principles (coloured version in the ebook).

6.8. Concluding Remarks

As was noted before and also becomes clear when reading the second chapter of this book and the case studies in Part 3, there is no unique view on inquiry-based

mathematics education (IBME). But the following broad conceptualisation of IBME of Artigue and Blomhøj (2013, p. 808) covers the perspectives of PLATINUM partners:

An educational perspective which aims to offer students the opportunity to experience how mathematical knowledge can be meaningfully developed. Thus, IBME becomes a powerful means of action, through personal and collective attempts at answering significant questions, making these experiences not just anecdotal but inspiring and structuring for the entire educational enterprise. As for IBSE,¹⁵ inquiry-based practices in mathematics involve diverse forms of activities combined in inquiry processes: elaborating questions; problem solving; modelling and mathematizing; searching for resources and ideas; exploring; analysing documents and data; experimenting; conjecturing; testing, explaining, reasoning, arguing and proving; defining and structuring; connecting, representing and communicating. These actions contribute to the students' knowledge and competences, but also to the formation of habits of mind for inquiry.

Artigue and Blomhøj (2013, p. 797) relate these actions to processes of inquiry of mathematicians and scientist:

Inquiry-based pedagogy can be defined loosely as a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work.

As we have seen, various theoretical frameworks support the conceptualisation of IBME and its implementation in practice. This diversity in the conceptualisation of IBME and supportive framework explains the diversity in the teaching units developed by PLATINUM partners. But they have one thing in common: all have been designed to promote conceptual understanding of mathematics through student inquiry. This means that in all teaching units the purpose of inquiry is to engage students deeply with concepts that they should learn or develop, in contrast with procedural learning or learning by rote. The concepts with which the students engage are already well-known and valued in mathematics and science, and have become essential ingredients of mathematical literacy. This contrasts with the purpose of inquiry for research mathematicians and scientists: they engage deeply with concepts to create *new* knowledge in their field of interest. Levy and Petrulis (2012) also distinguish between these purposes of inquiry and refer to them as *inquiry for learning*, when one explores what is already known, and *inquiry for knowledge building*, when the purpose is to build new knowledge. Most PLATINUM teaching units are aligned with inquiry for learning, in the form of guided or structured inquiry activities in which the lecturer acts as a facilitator of learning rather than as a source of information.

In addition, many PLATINUM teaching units have in common the use of ICT in inquiry activities. This, at first sight, is not surprising: mathematicians and scientists use ICT in inquiry and thus, if the goal is to let students work in ways similar as these professionals do, it is natural to let students use ICT as well. But there is an important difference: mathematicians and scientists use very sophisticated ICT tools that require deep knowledge of mathematics and their scientific discipline in order to use the tools successfully; most students lack the required mathematical and scientific knowledge and therefore need simpler ICT tools or a learning path for using the more sophisticated tools. The designers of tasks and teaching units in PLATINUM often use dynamic mathematics environments like GEOGEBRA and AUTOGRAPH to create for their students more dedicated and simpler tools for inquiry-based learning.

Important to the design of effective inquiry tasks and teaching units are the *three-layer model of inquiry* outlined in Chapter 2 (cf., Jaworski, 2019) and the notion of *community of inquiry* (CoI). Designs of mathematical activities for student inquiry improve when those involved have inquired into

¹⁵IBSE is an acronym of inquiry-based science education.

- the mathematical concepts with which students are supposed to engage in an inquiry way,
- IBME approaches to teaching and learning of the mathematical concepts, and into
- research findings of educators about IBME.

These three types of inquiry are often too much work for a single person and a community of inquiry is needed. Ideally such a community of inquiry comprised of different members in the field (e.g., discipline-based and/or general educational researchers, specialists in supporting students with identified needs, educational technologists, experts in mathematics and/or the field of application, students, etc.) so that shaping and implementing ideas for inquiry tasks can be taken to a higher level through collaboration of members of a CoI. Effort in task design is more sustainable when working in a team.

Sustainability of task design is fostered by documenting the work. Not only is documentation important for designers to keep track of discussions within the team and of design and implementation choices made, but it is important also for other lecturers who want to use or adapt tasks, or who simply want to be informed or inspired. For this reason we have included in this chapter tasks or task sequences developed by PLATINUM partners that exemplify design processes. The case studies presented in Part 3 are more detailed accounts of the partners' explorations of IBME at university level, and of their creation and use of teaching units for inquiry by their students. We hope that the case studies and this chapter on the design of inquiry activities inspire university lecturers to undertake similar explorations of IBME in their own practice.

References

- Ainsworth, S. E. (2008). The educational value of multiple-representations when learning complex scientific concepts. In J. K. Gilbert, M. Reiner & M. Nakhleh (Eds.), *Visualization: Theory and practice in science education* (pp. 191–208). Springer Verlag.
doi.org/10.1007/978-1-4020-5267-5_9
- Artigue, M., Batanero, C., & Kent, P. (2007). Mathematics thinking and learning at post-secondary level. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1011–1049). Information Age Publishing.
- Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM Mathematics Education*, 45(6), 797–810. doi.org/10.1007/s11858-013-0506-6
- Bereiter C. (1985). Toward a solution of the learning paradox. *Review of Educational Research*, 55(2), 201–226. doi.org/10.3102/00346543055002201
- Blum, W. & Leiß, D. (2007). How do students' and teachers deal with modelling problems? In C. Haines, P. Galbraith & W. Blum (Eds.), *Mathematical modelling: Education, engineering and economics* (pp. 222–231). Horwoord. doi.org/10.1533/9780857099419.5.221
- Blume, G. W., & Heid, M. K., (Eds.), *Research on technology and the teaching and learning of mathematics: Volume 1. research syntheses*. Information Age Publishing.
- Bosch, M., Chevallard, Y., García, F. J., & Monaghan, J. (Eds.). (2019). *Working with the Anthropological Theory of the Didactic in mathematics education: A comprehensive casebook*. Routledge. doi.org/10.4324/9780429198168
- Bransford, J. D., Brown, A. L., & Cocking, R. R. (Eds.). (2000). *How people learn: Brain, mind, experience, and school (Expanded edition)*. The National Academies Press.
doi.org/10.17226/9853
- Breen, S. & O'Shea, A. (2019). Designing mathematical thinking tasks. *PRIMUS*, 29(1), 9–20. doi.org/10.1080/10511970.2017.1396567
- Brousseau, G. (2002). *Theory of Didactical Situation in mathematics* (N. Balacheff, M. Cooper, R. Sutherland & V. Warfield (Eds. & Transl.). Kluwer Academic Publishers.
doi.org/10.1007/0-306-47211-2

- Bybee, R. W., Taylor, J. A., Gardner, A., Van Scotter, P., Powell, J. C., Westbrook, A., & Landes, N. (2006). *The BSCS 5E instructional model: Origins, effectiveness, and applications*. Biological Sciences Curriculum Study (BSCS).
<https://media.bsccs.org/bsccsmw/5es/bscs.5e.full.report.pdf>
- Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, 23(1), 2–33. doi.org/10.2307/749161
- Davis, R.B., Maher, C.A., & Noddings, N. (Eds.) (1990). *Constructivist views on the teaching and learning of mathematics*. National Council of Teachers of Mathematics.
- Donovan, M. S., & Bransford, J. D. (Eds.). (2005) *How students learn: History, mathematics, and science in the classroom*. The National Academies Press. doi.org/10.17226/10126
- Dorée, S. I. (2017). Turning routine exercises into activities that teach inquiry: A practical guide. *PRIMUS*, 27(2), 179–188. doi.org/10.1080/10511970.2016.1143900
- Dorier, J.-L. & Maaß, K. (2020). Inquiry-based mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 384–388). Springer Verlag. doi.org/10.1007/978-3-030-15789-0_176
- Eisenkraft, A. (2003). Expanding the 5E model. *The Science Teacher*, 70(6), 56–59.
- Escher, M. C. (1958) *Regelmäßige vlakverdeling [Regular division of the plane]* Stichting De Roos.
- Gómez-Chacón, I. M., & Kuzniak, A. (2015). Spaces for geometric work: Figural, instrumental, and discursive geneses of reasoning in a technology environment. *International Journal of Science and Mathematics Education*, 13(1), 201–226. doi.org/10.1007/s10763-013-9462-4
- Gómez-Chacón, I. M., Romero Albaladejo, I. M., & del Mar García López, M. M. (2016). Zig-zagging in geometrical reasoning in technological collaborative environments: A mathematical working space-framed study concerning cognition and affect. *ZDM Mathematics Education*, 48(6), 904–924. doi.org/10.1007/s11858-016-0755-2
- Goodchild, S., Apkarian, N., Rasmussen, C., & Katz, B. (2021). Critical stance within a community of inquiry in an advanced mathematics course for pre-service teachers. *Journal of Mathematics Teacher Education*, 24(3), 231–252. doi.org/10.1007/s10857-020-09456-2
- Heck, A. (2012). *Perspectives on an integrated computer learning environment* [Doctoral dissertation, University of Amsterdam]. <https://dare.uva.nl/record/409820>
- Heck, A. (2017). Using SOWISO to realize interactive mathematical documents for learning, practising, and assessing mathematics. *MSOR Connections*, 15(2), 6–16. doi.org/10.21100/msor.v15i2.412
- Heid, M.K., & Blume G.W. (Eds.), *Research on technology and the teaching and learning of mathematics: Volume 2. cases and perspectives*. Information Age Publishing.
- Holzcamp, K. (1995). *Lernen: Subjektwissenschaftliche Grundlegung*. Campus-Verlag.
- Holzcamp, K. (2013). Basic concepts of critical psychology. In E. Schraube & U. Osterkamp (Eds.), *Psychology from the standpoint of the subject: Selected writings of Klaus Holzcamp* (pp. 19–27). Palgrave Macmillan. doi.org/10.1057/9781137296436_2
- Hoyle, C., & Lagrange, J.-B. (Eds.). (2010). *Mathematics education and technology: Rethinking the terrain*. Springer Verlag. doi.org/10.1007/978-1-4419-0146-0
- Hughes-Hallett, D., Gleason, A M., Mccallum, W. G., Flath, D. E., Frazer Lock, P., Gordo, S. P., . . . , Tecsosky-Feldman, J. (2005). *Conceptests t/a calculus* (4th ed.). John Wiley & Sons Canada.
- Jaworski, B. (1994). *Investigating mathematics teaching: A constructivist enquiry*. Falmer Press. ERIC. <https://files.eric.ed.gov/fulltext/ED381350.pdf>
- Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, 9(2), 187–211. doi.org/10.1007/s10857-005-1223-z
- Jaworski, B. (2015). Teaching for mathematical thinking: Inquiry in mathematics learning and teaching. *Mathematics Teaching*, 248, 28–34.
- Jaworski, B. (2019). Inquiry-based practice in university mathematics teaching development. In D. Potari (Volume Ed.) & O. Chapman (Series Ed.), *International handbook of mathematics teacher education: Vol. 1. Knowledge, beliefs, and identity in mathematics teaching and teaching development* (pp. 275–302). Koninklijke Brill/Sense Publishers.
- Kaput, J. J. (1992). Technology and mathematics education. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515–556). Macmillan Publishing Company.
- Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C. (2018). Inquiry-oriented instruction: A conceptualization of the instructional principle. *PRIMUS* 28(1), 13–30. doi.org/10.1080/10511970.2017.1338807

- Kuster, G., Johnson, E., Rupnow, R., & Wilhelm, A. (2019). The inquiry-oriented instructional measure. *International Journal of Research in Undergraduate Mathematics Education*, 5(2), 183–204. doi.org/10.1007/s40753-019-00089-2
- Kwon, O. (2003). Guided reinvention of Euler algorithm: An analysis of progressive mathematization in RME-based differential equations course. *Journal of the Korean Society of Mathematical Education Series A: The Mathematical Education*, 42(3), 387–402.
- Laursen, S., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, 5(1), 129–146. doi.org/10.1007/s40753-019-00085-6
- Leung, A., & Baccaglini-Frank, A. (Eds.). (2017). *Digital technologies in designing mathematics education tasks. Potential and pitfalls*. doi.org/10.1007/978-3-319-43423-0
- Levy, P., & Petrucci, R. (2012). How do first-year university students experience inquiry and research, and what are the implications for the practice of inquiry-based learning? *Studies in Higher Education*, 37(1), 85–101. doi.org/10.1080/03075079.2010.499166
- Mason, J. (2002). *Mathematics teaching practice: A guide for university and college lecturers*. Horwood Publishing.
- Mason, J. (2008). From assenting to asserting. In O. Skovmose, P. Valero & O. R. Christensen (Eds.), *University science and mathematics education in transition* (pp. 17–40). Springer Verlag. doi.org/10.1007/978-0-387-09829-6
- Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd ed.). Pearson Education.
- Mason, J., & Johnston-Wilder, K. (2006). *Designing and using mathematical tasks* (2nd ed.). Tarquin.
- Mayer, R. E. (2020). *Multimedia learning* (3rd ed.). Cambridge University Press. doi.org/10.1017/9781316941355
- National Research Council (2000). *Inquiry and the national science education standards: A guide for teaching and learning*. The National Academies Press. doi.org/10.17226/9596
- Pedaste, M., Mäeots, M., Siiman, L., de Jong, T., van Riesen, S., Kamp, E., Manoli, C., Zacharia, Z., & Tsourlidaki, E. (2015). Phases of inquiry-based learning: Definitions and the inquiry cycle. *Educational Research Review*, 14, 47–61. doi.org/10.1016/j.edurev.2015.02.003
- Pointon, A., & Sangwin, C. J. (2003). An analysis of undergraduate core material in the light of hand-held computer algebra systems. *International Journal for Mathematical Education in Science and Technology*, 34(5), 671–686. doi.org/10.1080/0020739031000148930
- Rasmussen, C., & Blumenfeld, H. (2007). Reinventing solutions to systems of linear differential equations: A case of emergent models involving analytic expressions. *Journal of Mathematical Behavior*, 26(3), 195–210. doi.org/10.1016/j.jmathb.2007.09.004
- Rasmussen, C., Dunmyre, J., Fortune, N., & Keene, K. (2019). Modeling as a means to develop new ideas: The case of reinventing a bifurcation diagram, *PRIMUS*, 29(6), 509–526. doi.org/10.1080/10511970.2018.1472160
- Rasmussen, C., Keene, K., Dunmyre, J., & Fortune, N. (2018). *Inquiry oriented differential equations: Course materials*. <https://iode.wordpress.ncsu.edu>
- Rasmussen, C., & Kwon, O. N. (2007). An inquiry oriented approach to undergraduate mathematics. *Journal of Mathematical Behavior*, 26(3), 189–194. doi.org/10.1016/j.jmathb.2007.10.001
- Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education*, 37(5), 388–420. doi.org/10.2307/30034860
- Rogovchenko, Y., Rogovchenko, S., & Thomas, S. (2018). The use of nonstandard problems in an ODE course for engineers. In E. Bergqvist, M. Österholm, C. Granberg & L. Schuster (Eds.), *Proceedings of the 42nd conference for the Psychology of Mathematics Education* (Vol. 4, pp. 283–290). <http://hdl.handle.net/11250/2596252>
- Ross, K. A. (2013). *Elementary analysis: The theory of calculus* (2nd ed.). Springer Verlag. doi.org/10.1007/978-1-4614-6271-2
- Sáiz, E. (2020). *Geometría dinámica y teselaciones* [Dynamic geometry and tessellations]. www.geogebra.org/m/zvhyf6xj
- Sandoval, W. A., Bell, P., Coleman, E., Enyedy, N., & Suthers, D. (2000). *Designing knowledge representations for epistemic practices in science learning*. <https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.75.7691&rep=rep1&type=pdf>
- Scanlon, E., Taylor, Z. W., Raible, J., Bates, J., & Chini, J. J. (2021). Physics webpages create barriers to participation for people with disabilities: Five common web accessibility errors and possible solutions. *International Journal of STEM Education*, 8, article 25. doi.org/10.1186/s40594-021-00282-3

- Schattschneider, D. (1986). In black and white: how to create perfectly colored symmetric patterns. *Computer & Mathematics with Applications*, 12(3), 673–695. doi.org/10.1016/0898-1221(86)90418-9
- Schattschneider, D. (2010). The mathematical side of M. C. Escher. *Notices of the American Mathematical Society*, 57(6), 706–718.
- Schreffler, J., Vasquez III, E., Chini, J. J., & James, W. (2019). Universal Design for Learning in postsecondary STEM education for students with disabilities: A systematic literature review. *International Journal of STEM Education*, 6, article 8. doi.org/10.1186/s40594-019-0161-8
- Sfard, A. (2008). *Thinking as communicating*. Cambridge University Press. doi.org/10.1017/CB09780511499944
- Sfard, A., & McClain, K. (2002). Analyzing tools: Perspective on the role of designed artifacts in mathematics learning. *Journal of the Learning Sciences*, 11(2&3), 153–388. doi.org/10.1080/10508406.2002.9672135
- Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructive perspective. *Journal for Research in Mathematics Education*, 26(2), 114–145. doi.org/10.2307/749205
- Smith, G. H., Wood, L. N., Coupland, M., Stephenson, B., Crawford, K., & Ball, G. (1996). Constructing mathematical examinations to assess a range of knowledge and skills. *International Journal for Mathematical Education in Science and Technology*, 27(1), 65–77. doi.org/10.1080/0020739960270109
- Stein, M. K., Grover, B. W. & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, 33(2), 455–488. doi.org/10.3102/00028312033002455
- Swan, M. (2008). Designing a multiple representation learning experience in secondary algebra. *Educational Designer*, 1(1). www.educationaldesigner.org/ed/volume1/issue1/article3
- Treffert-Thomas, S., Jaworski, B., Hewitt, D., Vlaseros, N., & Anastasakis, M. (2019). Students as partners in complex number task design. In U. T. Jankvist, M. van den Heuvel-Panhuizen & M. Veldhuis (Eds.), *Proceedings of the eleventh Congress of the European Society for Research in Mathematics Education* (pp. 4859–4866). https://hal.archives-ouvertes.fr/hal-02459928
- Trouche, L. (2020a). Instrumentalization in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 392–403). Springer Verlag. doi.org/10.1007/978-3-030-15789-0_100013
- Trouche, L. (2020b). Instrumentation in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 404–412). Springer Verlag. doi.org/10.1007/978-3-030-15789-0_80
- van den Heuvel-Panhuizen, M., & van Zanten, M. (2020). Realistic Mathematics Education: A brief history of a longstanding reform movement. *Mediterranean Journal for Research in Mathematics Education*, 17(1), 65–73. doi.org/10.1016/j.cub.2014.12.009
- van Joolingen, W. R. & Zacharia, Z. C. (2009). Developments in inquiry learning. In N. Balacheff, S. Ludvigsen, T. de Jong, A. Lazonder & S. Barnes (Eds.), *Technology-enhanced learning: Principles and products* (pp. 21–37). Springer Verlag. doi.org/10.1007/978-1-4020-9827-7_2
- von Glaserfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press. ERIC. https://files.eric.ed.gov/fulltext/ED381352.pdf
- Watson, A., & Ohtani, M. (Eds.). (2015). *Task design in mathematics education*. Springer Verlag. doi.org/10.1007/978-3-319-09629-2
- Watson, A., Ohtani, M., Ainley, J., Bolite-Frant, J., Doorman, M., Kieran, C., . . . , & Yang, Y. (2013). Introduction. In C. Margolinas (Ed.), *Task design in mathematics education* (Proceedings of the ICM I Study 22). https://hal.archives-ouvertes.fr/hal-00834054
- Wenning, C. J. (2005) Levels of inquiry: hierarchies of pedagogical practices and inquiry processes. *Journal of Physics Teacher Education Online*, 2(3), 3–11.
- World Wide Web Consortium (2018). *Web Content Accessibility Guidelines (WCAG) 2.2*. www.w3.org/TR/WCAG/