Inquiry in University Mathematics Teaching and Learning: The PLATINUM Project
Corresponding author: Barbara Jaworski, B.Jaworski@lboro.ac.uk
`https://doi.org/10.5817/CZ.MUNI.M210-9983-2021-2`

CHAPTER 2

Conceptual Foundations of the PLATINUM Project

Barbara Jaworski, Inés M. Gómez-Chacón,
Reinhard Hochmuth

2.1. The PLATINUM Project

This chapter addresses the conceptual background underpinning PLATINUM, a project in the EU Erasmus+ programme. PLATINUM focuses on teaching and learning in university mathematics and particularly on IBME, Inquiry-Based Mathematics Education, involving mathematics teaching and learning and their development through the use of inquiry-based processes.

PLATINUM is a European (Erasmus+) project for the development of IBME in university education. Details of the project, that is, the partners, the concrete forms of cooperation, and so on, are described in Chapter 5 and on the PLATINUM website.[1] This chapter is about the common theoretical foundations of IBME and how they relate to different parts of the project and its origins, and to other chapters in the book.

PLATINUM stands for "**P**artnership for **L**earning **A**nd **T**eaching **IN** **U**niversity **M**athematics." Our partnership, within the EU Erasmus+ project, consists of eight teams of university mathematics lecturers, educators and researchers, in universities from seven European countries (see Chapter 5 for more details). Together, we form a partnership devoted to developing the teaching and learning of university mathematics that will enable university students' better understanding of mathematical concepts related to their programmes in mathematics, science, engineering, economics and other areas of study.

PLATINUM is characterised by the fact that the development of IBME and the project processes and practices are seen not as separate from each other, but as two strands that are analytically and theoretically distinct, but closely linked. Our proposal to the Erasmus+ programme included the following statement:

> Mathematics is a discipline central and foundational to many areas of study (including natural sciences, engineering, economics and teacher education) and to national success globally in academic prestige, business and trade, active citizenship and social entrepreneurship. Mathematics education in Higher Education influences the labour market and human lives, especially for people disadvantaged in educational opportunity, limiting their access to work and leisure; several *Intellectual Outputs* [IOs] [in accord with Erasmus+] emphasise this target group.
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> Mathematics can be experienced as difficult to learn and exclusive in terms of learning success. Considerable evidence shows that the learning of mathematics widely is highly procedural and not well adapted to using and applying mathematics in science

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[1] `https://platinum.uia.no`

> and engineering and the wider world; also, that students learn to reproduce mathematical procedures in line with tests and examinations, rather than developing a relational, applicable, creative view of mathematics that they can use more widely. We recognise a central need to enable all students to be conceptually focused with mathematics, to work with mathematics in creative and enterprising ways, and to equip them to apply mathematics in other disciplines and the world of work.
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> Therefore, we will develop an inquiry-based approach towards the teaching and learning of university mathematics and aim for the development of an international community of university mathematics lecturers who practice, explore and encourage others to use inquiry-based teaching approaches in teaching mathematics. These approaches will blend a range of ways of thinking, methods and technologies (including digital technology) in a well-balanced way to achieve more in-depth learning leading to meaningful application of mathematical concepts by our students. The needs of different groups of students will be in the focus of our activity, including those with special needs or other learning disadvantages.

Thus, the conceptual foundation of the PLATINUM project is *Inquiry-Based Mathematics Education* (IBME) and particularly the concept of *inquiry.* We have sought to develop an inquiry-based approach to mathematics teaching and learning at university level both theoretically and in our activity in eight universities in Europe.

The main purpose of inquiry is to engage those involved (students for example) *deeply* with concepts that they should learn or develop—in contrast with procedural learning or learning by rote—although, of course, following procedures or memorising facts or formulae can form a part of the learning process. Where mathematics is concerned, inquiry approaches in problems and tasks encourage students to get involved with the mathematics, not just using standardised rules and procedures but exploring/investigating processes and concepts, and trying to answer open-ended questions and solve problems. For their teachers/lecturers the challenge is to offer suitable problems/tasks through which their students' exploration can bring them to understanding the mathematics being presented to them in lectures. Of course, 'understanding' can mean different things for different people: Richard Skemp's (1976) position on *instrumental* versus *relational* understandings is well known; here we are rather thinking of understanding which is conceptual and relational. This challenge brings lecturers themselves into an inquiry process where their teaching of mathematics is concerned—conceiving suitable approaches to their students' engagement and bringing these into their practice with students.

In PLATINUM, we explore both didactic and pedagogic processes and practices and blend methods and resources to achieve development in teaching and learning. We utilise a developmental research approach in which partners 'walk the walk' of inquiry-based practice and share findings with others.

In this chapter we start by introducing the project briefly (above) and follow this with the reasons why new approaches to teaching and learning mathematics at university level are seen as important and necessary. We draw on relevant literature to situate our PLATINUM activity. The main part of the chapter (Sections 2.4 and 2.5) addresses our developmental approach in IBME from a PLATINUM perspective, drawing on inquiry in our seven countries, relevant literature, and explaining a three-layer theoretical model of inquiry which underpins the project. This model has acted as a framework for all our activity in PLATINUM, as we explain below. We introduce Intellectual Outputs (IOs), commensurate with an Erasmus+ programme, and discuss the activities in which we have engaged related to each IO, as a precursor to the chapters which follow.

A brief guide to the sections of this chapter follows: Section 2.2, *A Need to Redefine Teaching*, discusses some of the reasons why new approaches to teaching and learning are needed. Section 2.3, *IBME: A Brief History in the PLATINUM Countries and Beyond*, provides an outline of perceptions of IBME in the countries of PLATINUM. Section 2.4, *IBME in Mathematics Education*, presents international perspectives on which IBME is founded and as a basis for our work in PLATINUM. Section 2.5, *IBME in the PLATINUM Project*, discusses the theory of inquiry as it is used and developed within PLATINUM. In particular it introduces our Three-Layer Model of Inquiry on which PLATINUM is based and the key concepts of Inquiry Communities and Critical Alignment. Section 2.6, *Discussion and Conclusions*, concludes the chapter.

2.2. A Need to Redefine Teaching

Our focus in PLATINUM is the learning of mathematics of students in university level courses in a range of disciplines including mathematics, sciences, engineering, economics and so on. It is our overall aim that mathematics teaching should have the student in mind at all times, seeking to engender a student engagement that inspires deep levels of conceptual understanding, rather than only a superficial memorising of formulae and basic procedures. This is not to deny that a focus on formulae and procedures, or their memorisation, has its own value. Also, as we are aware, every mathematics didactic project proposal criticises in some sense the inadequate reality of existing mathematics teaching and especially the learning results. However, there is some consensus that understanding and relating mathematical concepts needs much more than memorisation and use of procedures, which is the basis of our proposed inquiry-based approach (cf., Alsina, 2002; Hawkes & Savage, 2000; Minards, 2013; Solomon & Croft, 2016; Treffert-Thomas & Jaworski, 2015). Of course, not every proposal is classified under the term inquiry. We ask, therefore, what is specific about the inquiry approach and always strive to emphasise this in our contribution.

We are aware, as the literature shows, that common practices in university learning and teaching leave many students with mathematical knowledge that does a disservice to mathematics and can be seen as inadequate for mathematical applications that depend on it (such as in the disciplines listed above (e.g., Faulkner et al., 2019)). Students themselves have reported dissatisfaction with what they are offered; for example, research into students' second-year experiences of mathematics courses in three UK universities showed many students disillusioned with their mathematics course. Solomon and Croft (2016) write:

> Student disengagement from undergraduate mathematics in the UK is widely reported ... raising basic questions as to how well-qualified students who report high levels of confidence and enjoyment at school can become so disillusioned with a subject which they have actively chosen to study at university. (p. 267)

It is in some sense common knowledge among professional colleagues that many students see the learning of mathematics as memorising formulae and procedures presented in lectures, that they expect to use in examination questions and thus contribute to their end-of-study grades and access to employment. Teachers often struggle to support students within the prevailing conditions. The following example points to a number of issues we face as university teachers:

> Recently, a colleague in linear algebra set a task that was formulated in such a way that it was recognisable that an already practised and known procedure would be useful to complete the task. But in order to implement this, it was first necessary to transform the task somewhat on a conceptual level in order to then apply the calculus. Technically, it was really only a small thing. But one had to have an idea of what it was all about.

> The result was quite bad because many students did not even get to the calculus part. This was compensated for by lowering the points required to pass. (For us, no more than 50-60% should fail. If more students do not pass the exam, there are follow-up questions, which one would like to avoid, also because the subsequent discussions are rarely productive).

There are many factors to consider, not least the culture and infrastructure of university education in which research takes academic precedence over teaching. Teaching is managed in lectures of several hundred students with exams designed to test what was presented in the lecture; and there is little time to support teaching development. Lecturers have typically teach in the ways they themselves were taught in university. For students the university teaching is very different from their experiences in school and lacks the kinds of guidance school provided. Solomon and Croft (2016) quote a response from one student who was asked how university mathematics differs from the school experience:

> It's sort of not as easy. 'Cos I used to find it easy then. I do like finding things out and getting the answers to things, but it's not as fun. So, I don't enjoy it . . . sometimes when I've just got an assignment back and it's awful, I just think 'Oh no, why am I doing this?' (p. 274)

Of course, we should not necessarily assume that when students say they like to *find things* out they mean what we might mean by the same words. For example, finding things out can consist of identifying and executing the respective correct calculation steps in a strictly prescribed scheme. However, we take this statement as illustrative evidence for students' needs in terms of their mathematical appreciation.

In the study volume from the 2000 conference of ICMI (International Commission on Mathematical Instruction) focusing on teaching and learning of mathematics at university level, Claudi Alsina (2001), a professor in mathematics from Spain, quotes US historian of mathematics Morris Kline, writing about the position of university mathematics professors:

> . . . appointment, promotion, tenure and salary are based entirely on status in research . . . but for most of the teaching that the universities are, or should be offering, the research professor is useless. (p. 3)

We might respond here that, since 1977, there have been new conditions, new insights, and new practices. However, we might also recognise some residual elements of Kline's words. Alsina himself (ibid) writes:

> There is a need to redefine mathematical research as a university activity, combining it with soundly-based teaching excellence. . . . Good teaching is according to a classic definition: "building understanding, communicating, engaging, problem solving, nurturing and organising for learning" a complete task that merits special attention and preparation (see Krantz, 1993). (p. 7)

It might be argued that, in the 20 years since the ICMI study, university teaching could have learned (and developed) internationally from what the study exposed and proposed: and to some extent we have. In the UK, for example, a government "Teaching Excellence Framework" evaluates universities on the quality of their teaching; most universities now include some generic courses for new lecturers on developing teaching. However, these generic courses are often found largely unhelpful for teachers of mathematics who claim they do not address teaching suited to mathematics itself (e.g., those related to symbolisation and proof). In Germany, for example, the three major mathematics associations (DMV, GDM and MNU) have requested that a corresponding recommendation on the subject-specific university didactic further training measures by a joint mathematics commission on the transition from school to university should be implemented.[2] In fact, some mathematics departments institute special courses for new lecturers in mathematics, to address concepts seen as directly related to mathematics teaching (see for example Winsløw et al., 2021). It remains true however that, despite such innovation, mathematics teaching at university level can benefit from further development. We are aware, of course, that not every proposal for development is classified under the term inquiry. We ask, therefore, what is specific about the inquiry approach and always strive to emphasise this in our contribution.

In PLATINUM, we have addressed the idea of development based in inquiry processes involving both lecturers and students, as we address below. This development has taken place in eight universities in seven countries in Europe, each with its own language and culture, its own higher education structure and university systems, and its own ways of approaching mathematics teaching and learning. In Section 2.3 we provide some historical information relating to IBME in these countries.

2.3. IBME: A Brief History in the PLATINUM Countries and Beyond

PLATINUM partners come from seven countries in Europe: the Czech Republic, Germany, the Netherlands, Norway, Spain, Ukraine and the United Kingdom. Details of the educational systems and specifically of university education in mathematics can be found on the PLATINUM website[3] and in the proposal to Erasmus+, also on the website.

Here we focus specifically (and in outline only) on the history and development of IBME in the countries of PLATINUM as experienced by PLATINUM colleagues. This experience relates fundamentally to *who we are* in our national situations and our personal teaching-learning activity. To some degree, all of us teach mathematics to university students in university courses. This might involve courses in mathematics for mathematics students, students of engineering or science, of economics, medicine and so on. Some of us teach prospective teachers of mathematics. Some are mathematicians, developing knowledge in mathematics through their research; some are mathematics educators, researching many aspects of teaching, learning and development in the didactics and pedagogies of mathematics. It is this latter group that has most experience of IBME through their need to study the literature of mathematics education including its history and development.

The theory(ies) behind IBME develop from some eminent educationalists and mathematicians in our history. For example, John Dewey (1859-1952), University of Chicago, and George Polya (1887–1985), Stanford University, were significant forebears to whom we can trace many of the aspects of active learning in general and IBME in particular. In our countries, we refer to significant pioneers of problem solving in mathematics, Hans Freudenthal in the Netherlands, Miguel Guzman in Spain, Erik Wittman in Germany, John Mason in the UK; Alan Schoenfeld in the US is well-known internationally and a frequent visitor to Europe. We say more about their influence in Section 2.5.

In PLATINUM, with our central focus on IBME, we are all aware of a number of high-profile European research projects into the teaching and learning of mathematics (and often of science as well) from *inquiry-based* principles, mostly at primary and secondary school levels. Colleagues at BUT in the Czech Republic point to the Fibonacci Project (Large scale dissemination of inquiry-based science and mathematics education), the PROFILES project (Professional Reflection-Oriented Focus on Inquiry-based Learning and Education through Science), and the project MaSciL

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[2] `http://mathematik-schule-hochschule.de/images/Massnahmenkatalog_DMV_GDM_MNU.pdf`

[3] `https://platinum.uia.no`

(Mathematics and Science for life).[4] Research into practices of teaching and learning mathematics in schools has permeated all of our countries, with colleagues who are involved in teacher education being the researchers alongside school teachers.

For example, colleagues at UvA in the Netherlands point to the major research institute on mathematics education, the Freudenthal Institute (FI), [initiated by Hans Freudenthal (1905–1990)] which had until recently researched only in primary and secondary schools. They write:

> This is reflected in the European projects in which FI members participate(d): Fibonacci, PRIMAS, MaSciL, MERIA, and TIME. The conceptual framework for the work of the mathematics education researchers at FI always embeds Design Research and Realistic Mathematics Education (RME). It is based in engaging students in realistic (to them) problems which might be *real world problems*, perhaps involving *modelling*, or *mathematical problems* that are 'real' for the students who try to solve them.

In Mathematics Education, developmental work in the Netherlands based on RME is well known and frequently emulated internationally. For example, colleagues at UCM in Spain write, "some universities in Spain offer a conceptualisation of IBME linked to the theory of Realistic Mathematics Education (RME) (Gravemeijer & Doorman, 1999; Alsina, 2002)." They claim that "reality-based problems, as mathematical objects, promote initially a model that is context-specific. Their affordances are substantially different from those offered by the problem-solving approach."

The 'Problem-Solving Approach' refers to research and development into the use of (mathematical) problems as an introduction to mathematics learning and teaching in classrooms. Colleagues at UCM in Spain refer to "a long tradition of research and practice in our field going back to the seminal work of George Polya (e.g., Polya, 1945). Miguel de Guzmán, professor at UCM and president of the International Council of Mathematics Instruction (ICMI), encouraged teaching and learning at university level in this direction by publishing various books and developing a theoretical framework that plays an essential role in the solving of problems. The teacher training programmes under this approach were promoted with the support of the Spanish Ministry of Education and with the collaboration of international experts such as Schoenfeld (USA) who was invited to give courses and lectures."

In university education in Germany, problem solving in mathematics is seen as a specific competence, which is generally conceptualised along the lines of Polya (1945). With regard to mathematical learning processes at school as well as at university, problem-solving is considered important, especially with regard to multifaceted heuristics when working on tasks and problems, and is taught accordingly in order to make corresponding experiences possible (Bruder & Collet, 2011). However, it is assumed that the adoption of this competence in the context of the acquisition of the new and abstract material of Analysis and Linear Algebra is too big a hurdle for many students in their first semester. This is one of the reasons why this competence should initially be acquired in a special course which, in terms of mathematical content, focuses much more on school mathematics and ties in with it. With regard to teaching profession students, this has the welcome additional effect that they can acquire in-depth school knowledge on some topics. It is assumed that the problem-solving competence

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[4] The Fibonacci Project—Large scale dissemination of inquiry-based science and mathematics education (`www.fibonacci-project.eu`); the MaSciL Project—Mathematics and Science for Life (`https://mascil-project.ph-freiburg.de`); the MERIA Project—Mathematics Education Relevant, Interesting and Applicable (`https://meria-project.eu`); The PRIMAS Project—Promoting Inquiry in Mathematics and Science Education Across Europe (`https://primas-project.eu`); The PROFILES project (`www.profiles-project.eu/`); the TIME Project (`https://timeproject.org`).

acquired in these courses can then be used in the context of the more abstract requirements of the classical lecture. One of the first such courses was established by Grieser (2018) at the University of Oldenburg. Other universities have subsequently established similar courses (Hochmuth et al., 2022). Such and related approaches have been attempted in recent years, especially in preparatory and bridge courses. A good overview of this is provided by the practical examples presented in (Biehler et al., 2021) from the Competence Centre for Higher Education Didactics in Mathematics.[5]

Although the European projects mentioned above, as well as RME and The Problem-Solving Approach, focused (mainly) on mathematics learning and teaching *for school students*, nevertheless, researchers from universities often led the work in these projects. These researchers were usually employed in mathematics education, perhaps in teacher education, whereas teachers of mathematics (at university level) are less likely to be involved in such research. However, it is not always so clearly distinguished. Colleagues in the Ukraine write:

> We believe that IBME refers to a student-oriented paradigm for mathematics and science teaching, in which students are invited to work in ways similar to mathematicians and scientists. The best teachers and lecturers used problem-based learning, solving research and applied problems, the case method, and the implementation of group projects in order to stimulate pupils or students to search, to conscientious and, if possible, independent construction of knowledge, thereby achieving understanding, and not formal memorisation. Although the term (IBME) was not literally used in the Ukrainian scientific community, (university) teachers, often intuitively, used certain approaches that are characteristic of IBL (posing research questions, formulating and testing/proving hypotheses, etc.).

In Norway, the national Centre for Research, Innovation and Collaboration in Mathematics Teaching[6] (MatRIC) was established to focus on mathematics teaching at university level. Researchers in MatRIC had conducted a survey of Norwegian university mathematics teachers and one colleague wrote:

> The survey focused on active learning approaches rather than IBME. My interpretation is that it does not reveal much about the incidence of IBME. As far as I am aware, IBME is more of a topic of discussion between [university] mathematics educators and lecturers, there may be some small pools of activity—for example [one colleague] developed some interesting blended learning approaches (not specifically IBME), and these were researched and reported in a PhD study and in a paper in IJRUME (International Journal for Research in Undergraduate Mathematics Education). However, these innovations came to an end when the class was incorporated into the larger cohort of first year engineering students. The mathematics team at [location], have, last semester, tried to introduce a modelling project into the first semester mathematics course. There is some intersection between this and the notion of IBME, but it was not an effort specifically designed to introduce or develop IBME, I really do not know what is happening in other institutions [in Norway], and my feeling is that there is very little substantial development of IBME approaches implemented at [university level].

In the UK, there is a history dating back to the 1960s of ‘investigational activity’ or ‘investigations’ in mathematics, often deriving from workshops and conferences of the Association of Teachers of Mathematics (ATM), and promoted by teachers in Colleges and Department of Education (Jaworski, 1994). An influential figure was Caleb Gattegno, who had written in 1960:

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[5] `www.khdm.de`

[6] `www.matric.no`

> When we know why we do something in the classroom and what effect it has on our students, we shall be able to claim that we are contributing to the clarification of our activity as if it were a science.

Gattengo influenced the establishment of the Association of Teachers of Mathematics (ATM) and many publications offering starting points for explorative activity in mathematics by students, and advice for teachers. Such activity was described as follows:

> In contrast to tasks set by the teacher—doing exercises, learning definitions, following worked examples—in mathematics activity the thinking, decisions, projects undertaken were under the control of the learner. It was the learner's activity. (Love, 1988, p. 249).

While such 'activity' related mainly to school classrooms, it was promoted for university students (often school teachers) studying with the Open University in the UK, through Polya's (1945) book, "How to Solve It," developing problem-solving heuristics, and through the work of John Mason and colleagues who presented problem-solving heuristics in a book "Thinking Mathematically" (Mason et al., 2010). In the US, at this time, the problem-solving movement based on Polya's work led to research in classrooms studying students' problem-solving activity and, in particular the developing thinking of the teachers involved (e.g., Cobb et al., 1990), In the UK, in parallel, a study focused on teachers developing their use of investigational activities with students led to a recognition of teacher inquiry in the development of mathematics teaching (Jaworski, 1994) revealing issues and tensions experienced by the teachers and professional growth emerging from the activity and research. Although the term IBME was not used in the UK at this time, the ideas of inquiry in mathematics problem solving (for students at all levels) and in teachers' explorations in teaching, revealing issues and tensions for teachers, can be seen as strongly related to the IBME approaches employed in PLATINUM. These have been compared to the research approaches used by mathematicians in exploring beyond current mathematical knowledge and opening up new vistas in mathematics (e.g., Burton, 2004).

The fact that research-based learning has again become a focus of discussion, especially since the 1990s, is due to the context of "Bologna." At least in Germany, conflicting moments are seen here:

> One of the goals of these reforms was that studies should increase the general employability of university graduates. To this end, in addition to subject-specific competences, more general skills were to be taught, which were referred to as key competences. Problem-oriented forms of learning, project-oriented and also research-based learning were identified as conducive to this. (Huber & Reinmann, 2019, p. 22, translation by the authors)

Indeed, this development was complemented in Germany by an increased competitive orientation in funding policy, e.g. through the "Excellence Initiative," which encouraged universities to support research-based learning in teaching. However, the universities' public commitments to research-based learning concealed very different degrees of actual preparation, promotion and coordination of such projects (Huber & Reinmann, 2019, p. 23). Following the principles of "New Public Management," quality standards and measurements, formative and summative evaluations including statistics on student success and student evaluations of courses were introduced (cf., Wildt, 2013, p. 37) and accompanied the introduction and implementation of the projects. This in particular was accompanied critically at an early stage, e.g. with regard to inadequate content specifications, and the danger of a "didacticisation" of the university to the detriment of its scientific character was seen (cf., Mittelstraß, 1996). If one takes a look at mathematics-related initiatives for the implementation of

IBME projects, as will be done in the following, it is noticeable that these connections (which have just been hinted at) are largely ignored: The idea that students should learn concepts in depth, for example, is taken up, but without problematising the socio-institutional context and the contradictory teaching-learning conditions that go along with it. So the question arises as to whether IBME can work in this way and achieve the objectives associated with them. With regard to this issue, the largely unanswered and, under the given boundary conditions, possibly unsolvable problem of the examinations is a striking symptom, at least for Germany. With regard to teaching and learning in schools, such questions were for example analysed systematically in (Holzkamp, 1995). These analyses are taken up in Chapter 14 and discussed in more detail with regard to the concepts underlying PLATINUM.

2.4. IBME in Mathematics Education

At its simplest, inquiry involves exploring, investigating, asking questions and solving problems. In mathematics, this includes exploring mathematical relationships, investigating mathematical conjectures, asking questions about mathematical applications and solving problems in mathematics and the wider world. Artigue and Blomhøj (2013) write:

> Inquiry-based pedagogy can be defined loosely as a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work. (p. 797)

All mathematicians who do research in mathematics are familiar with the processes of inquiry, since mathematics itself, as a discipline, progresses through inquiry. For example, in 1997, the mathematician Andrew Wiles, provided a solution for the long-unsolved problem, known as ‘Fermat’s last theorem’, posed by Pierre de Fermat a French mathematician of the seventeenth century. This achievement is described in the Foreword to Simon Singh’s book (1997) addressing the proving of the problem, as “the Himalayan peak of number theory.” Singh provides a gripping account of the inquiry process engaged in by Andrew Wiles (Singh, 1997). Quoting Wiles, he writes:

> I used to come up to my study, and start trying to find patterns. I tried doing calculations which explain some little piece of mathematics. I tried to fit it in with some previous broad conceptual understanding of some part of mathematics that would clarify the particular problem I was thinking about. Sometimes that would involve going and looking up in a book to see how it’s done there. Sometimes it was a question of modifying things a bit, doing a little extra calculation. And sometimes I realized that nothing that had ever been done before was any use at all. Then I just had to find something completely new – it’s a mystery where that comes from. (Wiles, in Singh, 1997, p. 227–228)

However, unlike the inquiry processes of researching mathematics, the processes of teaching mathematics over the centuries, and particularly in current times, have largely avoided the inquiry involved in research: they have presented mathematics as a top-down deductive process, explaining, justifying, and offering procedures to be learned, often unrelated to the inquiry processes that underpinned them. As we have mentioned above, a result of this teaching approach has often been that students memorise the presented results of the research, without understanding the underlying concepts, and consequentially struggle to apply mathematics and solve problems (see for example Alsina, 2001). Of course, “teaching approach” does not include only what happens in the classes but also the didactics and pedagogy behind what is done.

In schools, pedagogical considerations are naturally strongly oriented towards the organisation of lessons and the role of teachers: research-based work is rather difficult to squeeze into 45’ or maximum 90’ units. Questions and problems posed must be able to be worked through quickly, and also in such a way that possible answers or questions from the students do not overwhelm the teacher. The role that is usually attributed to teachers is that they are held responsible for the acquisition of knowledge by the students. However, research-based learning, by its very nature, must include failure, just as research does. Research without failure is not possible. A lesson in which questions are raised that teachers cannot answer, or at least cannot deal with directly, reflects badly on the teachers. Here, too, a change in thinking is necessary: A lesson in which students ask questions that are not trivial and the teacher therefore cannot answer immediately is a good lesson! Added to this is the constantly envisaged assessment of students' performance. Poor performance and assessments must be avoided. If this becomes the main goal of the pupils' activity, the content aspect of what is taught and to be learned recedes into the background, becomes secondary. A corresponding pedagogy that aims at explorative learning on the matter at hand must therefore be at least partially unassessed. However, small steps in the development of the subject matter and the best possible control over the pupils' actions do not only dominate mathematics lessons, but also the other subjects. Here, too, a change in pedagogy would be a desirable goal. Responding to an inquiry-based research project in Norway, Skovsmose and Säljö (2008) contrast inquiry-based teaching with what they called "The Exercise Paradigm":

> This [the exercise paradigm] implies that the activities engaged in the classroom to a large extent involve struggling with pre-formulated exercises that get their meaning through what the teacher has just lectured about. An exercise traditionally has one, and only one, correct answer, and finding this answer will steer the whole cycle of classroom activities and the obligations of the partners involved. (p. 40)

They suggest that inquiry-based practice takes us beyond the exercise paradigm:

> The ambition of promoting mathematical inquiry can be seen as a general expression of the idea that there are many educational possibilities to be explored beyond the exercise paradigm. (p. 4)

Some years earlier, Hiebert and colleagues in the US (1996) wrote:

> We argue that reform in curriculum and instruction should be based on allowing students to problematise the subject. Rather than mastering skills and applying them, students should be engaged in resolving problems. (p. 12)

Addressing the historical roots for inquiry and building on (Hiebert et al., 1996; Artigue & Blomhøj, 2013) attribute the concept of inquiry to John Dewey (1938), particularly his contribution in developing 'reflective inquiry' to form a basis for a pedagogical practice. They write:

> Dewey (1938) sees learning as an adaptive process in which experience is the driver for creating connections between sensations and ideas, through a controlled and reflective process, labelled reflective inquiry. The organization of students' experience and the development of general habits of mind for learning through reflective inquiry is thus an essential function of education. (p. 798)

Dewey (1933) himself has written:

> . . . reflective thinking, in distinction to other operations to which we apply the name of thought, involves (1) a state of doubt, hesitation, perplexity, mental difficulty, in which thinking originates, and (2) an act of searching, hunting, inquiring, to find material that will resolve the doubt, settle and dispose of the perplexity (p. 12) . . . Demand for the solution of a perplexity is the steadying and guiding factor in the entire process of reflection. (p. 14)

The concept of reflective inquiry and the language of Dewey here capture well the perspectives and approach we have taken to our work in PLATINUM that we have called *inquiry-based education.*

Particularly, also because of our focus on university mathematics, we see the need to complement Dewey's approach somewhat: As Artigue and Blomhøj (2013) point out, referring to Bachelard's concept of epistemological obstacles, successful IBME requires the "careful organisation of students' experience and inquiries to allow them to face the limitation of common sense," which again emphasises the teacher's responsibility from a more content-related point of view. Also the notion of students' acquisition of general discovery and problem-solving competences must always be complemented by concrete content-related activities that promote "incorporating local constructs into more regional perspectives" (Artigue & Blomhøj, 2013, p. 800).

We mention other sources highly relevant to our work in PLATINUM. Cochran Smith and Lytle (1999), working with teachers in the US, developed a construct that they call "Inquiry as Stance." This captures the ways in which teachers took on a mantle of inquiry-based practice and 'made it their own;' it relates strongly to the idea of "Inquiry as a Way of Being" (Jaworski, 2004), used in our PLATINUM model (see below). Gordon Wells (1999) wrote about "Dialogical Inquiry," in which teachers (or students) engage together in dialogue that is inquiry-based. Alrø and Skovsmose (2002) have also related dialogue and inquiry in mathematics classrooms, writing of the critical nature of inquiry in "landscapes of investigation" (see also Skovsmose & Säljö, 2008). Wells (1999) talks specifically of *communities of inquiry* through which dialogue encourages concept formation. Here again, this is highly relevant for PLATINUM.

In much of the literature cited above, authors have discussed aspects of inquiry-based practice with relation to teaching and learning in school classrooms. While we might see inquiry-based practices offering didactics and pedagogy desirable for students' deeper engagement with mathematics, there are factors in both learning and teaching within educational infrastructures that promote less desirable activity and outcomes. For example, in schools, classes of 30 students taught by one teacher might favour prescriptive teaching and learning, they also leave little responsibility for the students, protecting them from taking initiative and thinking for themselves. When students who have always been nurtured in such ways then go on to university, it is not uncommon for them to wish to be taken more by the hand and, understandably, to expect that the learning they have experienced at school will also be successful at university. That the university needs to organise a pedagogical transformation process is not well understood. For example, for the student, the concept of lecturing to large groups of students (often several hundreds in one lecture theatre) can look very different from teaching in classrooms of 30 students. Simply holding lectures with difficult content and expecting students to swim their way through them doesn't work for many students. The expansion of many small-step examinations in this situation and the offer of very tightly guided assistance, while accommodating the students and showing "success," achieve the opposite of what they actually want in pedagogical terms, in particular no transformation towards inquiry, or research-based learning.

It is fair to say that, while a wealth of literature addresses development in mathematics learning and teaching at school level, often promoting processes based in inquiry, there is yet correspondingly little at university level. The transformation to inquiry-based teaching and learning builds on similar motives and theoretical reasoning at both school and university levels, but, at the practical level, has to take account of differences in both culture and infrastructure. While many aspects of IBME have the same meanings and relevance, their application in university teaching can look different from their application in school classrooms.

Although yet small in comparison with school-based education, a literature relating to teaching and learning mathematics in Higher Education (HE) is growing. For example, in 1998, in Singapore, the 7th study conference in the ICMI study series was held, focusing on *the teaching and learning of mathematics at university level* (see the study volume, Holton, 2001). In the second edition of the Encyclopedia of Mathematics Education (Lerman, 2020), we see entries on *University Mathematics Education, Teaching practices at University Level and Preparation and Professional Development of University Mathematics Teachers* (see pp. 670–674). A chapter on *Research on University Mathematics Education* (Winsløw et al., 2018) is included in a book *Developing Research in Mathematics Education: Twenty Years of Communication, Cooperation and Collaboration in Europe* (Dreyfus et al., 2018). The INDRUM conference (International Network for Development of Research in University Mathematics—an offshoot of CERME, Congress of the European Society for Research in Mathematics Education) has, in 2020, had its third conference. A book from INDRUM (Durand-Guerrier et al., 2021) has been published. Although IBME-related activity is touched-on in some of these sources, there is as yet very little published that specifically addresses IBME in HE. We hope that our sources from PLATINUM will stimulate the beginnings of an IBME corpus of research and development in Higher Education.

As with the differences in growth of literature relating to school and university mathematics learning and teaching, it is also the case that professional development programmes for university teachers are less common and less well-developed than those compulsory for school teachers. Although many universities in different countries have their own non-subject-specific professional development programmes there is much less emphasis on subject teaching development (see for example Winsløw et al., 2021). In the PLATINUM project we have taken seriously the need to think about teacher education for IBME teaching and learning, rooted in the motives and principles introduced above. In Section 2.5, we address these motives and principles, and the sorts of practices that we have developed over the time of our project.

2.5. IBME in the PLATINUM Project

The conceptual foundation of the PLATINUM project is an *inquiry-based* approach to mathematics teaching and learning, recognising *inquiry* very much in the way expressed by Dewey as '*reflective inquiry*' (above). In this project we explore (inquire into) both didactic and pedagogic processes and practices, and blend methods and resources to achieve development in mathematics teaching and learning. We utilise a developmental research approach in which partners 'walk the walk' of inquiry-based practice, share findings with others and look critically at what they are doing as they do it.

The main purpose of *inquiry* is to engage our students *deeply* with concepts that they should learn or develop, in contrast with procedural learning or learning by rote. Where mathematics is concerned, inquiry approaches in problems and tasks encourage students to get involved with the mathematics, asking and trying to answer questions, and exploring/investigating processes and concepts in contrast with an exercise paradigm as mentioned above. For their teachers/lecturers the challenge is to offer problems/tasks such that, through inquiry-based engagement, their students can come to understand the mathematics being presented to them in lectures. This challenge brings lecturers themselves into an inquiry process where their teaching of mathematics is concerned—conceiving suitable approaches to their students' engagement and bringing these into their practice with students.

Within our PLATINUM partnership, *collaboration* has been at the heart of our activity: it has been our intention to form *Communities of Inquiry* (CoI) at university level. In fact, the entire partnership can be regarded as one large community of inquiry. Together we inquire into the processes of learning and teaching with which we engage; we design inquiry-based tasks for our students and associated teaching units related to the mathematical topics we teach. Moreover, each partner group itself constitutes an inquiry community. Its members work together within their particular system and culture (both mathematical and educational) and the ways of working and underlying assumptions that this entails. Part of our inquiry process involves inquiring into and becoming more aware of the ways of thinking and reasoning which underpin our educational activity in mathematics whether in large lectures or other kinds of grouping. We ask *why* things are the way they are, and whether there are alternatives that might be more effective for our students' learning of mathematics. We explore possibilities and use inquiry where we can, looking critically at practices and their outcomes, both established and innovative.

It seems fair to say that an inquiry community, in the PLATINUM context, means a group of people (lecturers and researchers) who sincerely explore possibilities for the use of inquiry-based activities with our students to achieve students' more conceptual understandings of mathematics. In so doing, we hope to create teaching units in which students explore mathematical questions and work together to discuss their inquiry processes and argue potential solutions. Through such practices, we hope that students, together, will develop their approaches towards exploring and solving problems, feel the satisfaction of understanding what they are doing and why they are doing it, and will enjoy their mathematics. We therefore extend our concept of inquiry community to the students themselves.

We recognise that the transition from a more traditional to an inquiry-based approach to teaching and learning mathematics in not easy. It is not easy for the students who have been enculturated into a top-down didactic approach in which their involvement is to listen, make notes, learn and reproduce theorems and their proofs, and tackle stereotypical problems. In fact, many students resent being asked to explore mathematical ideas rather than being told exactly what to learn and how to solve. It is thus also not easy for the teachers to encourage their students to work in these new ways, especially considering that the teachers themselves are not so familiar with what an inquiry approach can mean. The transition can destabilise the entire *didactic contract* between teachers and their students (see for example Alsina, 2001; Brousseau, 2002). A didactic contract creates (often implicitly) the expectations of teacher and students of each other: in more traditional forms of teaching, the teacher has responsibility to tell students what they have to learn and understand, and to guide them in ways of achieving what is required. Students have responsibility to work in the ways guided by the teacher, to make effort to learn what is presented and to become skilled in working on tasks and solving exercises. In inquiry-based activity, it is the student's responsibility to work on a task, asking questions, seeking patterns, making conjectures, proving or disproving. The teacher's responsibility is to design tasks through which students' activity may reveal key concepts and relationships to progress mathematical understanding. Inquiry activity is far less well-defined than traditional activity and, when new to students and teacher alike, can present difficulty and confusion.

In PLATINUM, therefore, we both recognise the difficulties of transition, towards inquiry, and seek to identify their nature and outcomes. Recognising that we have set out to undertake practices which are both demanding and challenging, we reflect on our practices to identify issues and tensions in what we are seeking to achieve and the particular outcomes of what we do (following Dewey, quoted above). By identifying the challenges and sharing with our colleagues in our inquiry communities, we come to know more about the inquiry-based processes we seek to engage. We learn what it means to work in inquiry ways in practice as well as to adapt in theory or belief. Thus, we walk the walk as well as talking the talk! In Part 3 of this book each chapter offers a case study from one of the partners in PLATINUM. These cases document the thinking and experiences of the partners. Of course, each one relates to the situation and culture in which it takes place, so together they offer a panorama of inquiry-based university mathematics development in Europe.

We have written above about two 'layers' of inquiry: that is inquiry in mathematics between teachers and students, and inquiry in teaching as teachers explore new approaches to teaching and learn from their own reflective inquiry. Thus, teachers within the project seek, not only, to engage with their students in inquiry in practice, but also, to provide an account of their activity, the outcomes, issues and tensions arising from it, that they can share with other practitioners beyond the project (for example, in the cases of Part 3 of the book). The process here is developmental. As teachers participate in inquiry activity, learning from their engagement with students, their practice develops.

This indicates a third, developmental layer in which reflection and analysis provide insights into the processes and practices of inquiry and their outcomes. What we see here is what we call developmental research. Here we raise questions related to our practice and its development and address these questions in a systematic way: our aim is to provide insights into aspects of mathematics learning and teaching and the issues and tensions that arise for us as practitioners and researchers. This creates a new layer in our model. Thus, our inquiry model—see Figure 2.1, developed from (Jaworski, 2006, 2019)—constitutes three layers as follows.

Inquiry in:

1. engaging with mathematics in inquiry-based teaching-learning situations with students; students will engage with inquiry in mathematics with the aim of developing their own in-depth understanding of mathematical concepts.
2. exploring teaching processes, the didactics and pedagogies involved in student inquiry, and their use in teaching-learning situations to achieve desired student outcomes; teachers/lecturers will reflect on their own practices and seek to understand better the teaching approaches that enable their students' understanding.
3. the entire developmental process in which participants reflect on practices in the other two layers, and gather, analyse, and feed-back data to inform practice and develop knowledge in practice. Communities of inquiry will both enable and progress this development and share with others beyond the project. Sharing of insights gained and issues and tensions addressed is an important vehicle for promoting development more widely.

These layers are deeply interrelated. Teachers/lecturers, inquiring into their teaching, focus centrally on their students' learning through inquiry. Teacher-researchers, reflecting and analysing data from the other two layers, feedback what is learned to the practices they are in the process of developing. The whole constitutes an interrelated developmental process represented by the figure below.

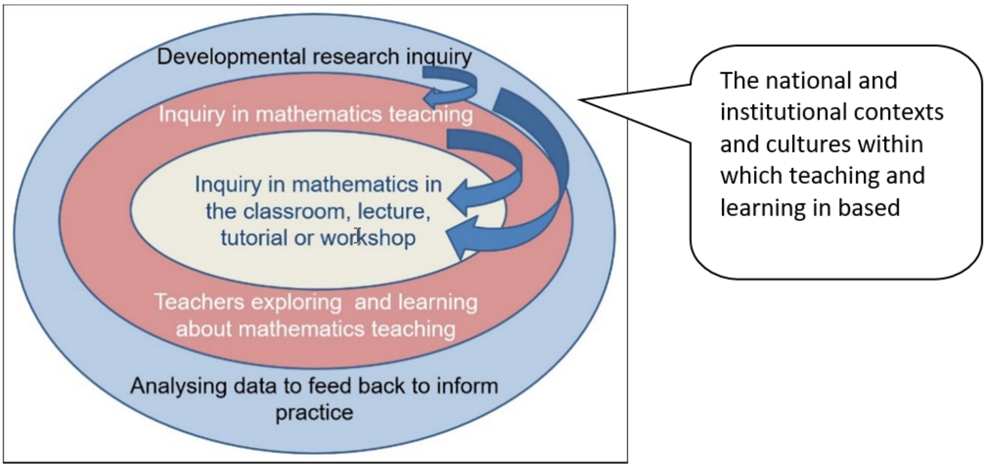

FIGURE 2.1. The Three-Layer Model of Inquiry.

Important to the inquiry process in its three layers, is the concept of 'community'. An *inquiry community* in PLATINUM consists of lecturers and/or students working together in inquiry ways to achieve learning and development. Central to our model is the belief that engagement with others, into concepts we seek to learn/develop, enriches engagement and provides opportunity for individuals to broaden their own thinking and to clarify their own conceptions. This provides opportunity for colleagues to look critically at the practices in which they engage, and to introduce and explore changes to practice. Such a critical approach within a supportive environment enables participants to address problems and issues with teaching and learning which might otherwise be beyond individual resolution. In the project, we expect our analysis of data from our own teaching practices to allow us to report on the outcomes of our activity with evidence to support what we claim regarding our developments in teaching-learning through inquiry.

Associated with the important concept, *Community of Inquiry* (CoI), is a concept we call *Critical Alignment* (e.g., Jaworski, 2006): While following community norms and expectations, we look critically at outcomes and achievements; we question, or inquire into, alternative practices as part of our engagement. These two concepts are strongly related to Etienne Wenger's (1998) *Community of Practice* (CoP). Wenger postulates that a Community of Practice builds on three constructs: *mutual engagement*, *joint enterprise*, and *shared repertoire*. In our developmental practice, we engage with each other in the agreed practices of our community and with the same ways of being and doing. However, we also start to introduce new practices/activities and to explore their development. Wenger conceptualises 'belonging' to a CoP as involving *engagement*, *imagination*, and *alignment*: *Belonging* entails engaging with others, using *imagination* in forging our own trajectories in the practice, and *alignment* with the norms and expectations of the practice. Thus, our practice changes as we engage with and explore the new elements, side-by-side with established practices. Thus, a Community of Inquiry (CoI) can be seen to encompass *mutual engagement*, *joint enterprise*, and *shared repertoire* as in a CoP, and *belonging* to a CoI to involve engagement and imagination as in a CoP. The main point of difference between CoP and CoI, lies with *alignment*. Simply aligning with the norms and expectations of a practice can result in perpetuation of elements of the practice that are not what we would like to see. It is very difficult, often, not to align with existing practices (the ways in which all our colleagues are engaged), but, while we align we can be aware of the need for change, and seek out ways of achieving the changes we would like to see: looking critically at what we are doing as we do it, with our eyes on possibilities for change. Thus, in an inquiry community, it is proposed that *belonging* presupposes an element of *critical alignment*: While following community norms and expectations, we look critically at outcomes and achievements, and question, or inquire into, alternative practices as part of our engagement. This concept of critical alignment can be seen as central to developmental activity in PLATINUM; indeed, one partner group uses the concept to present activity in their case study (see Chapter 14).

In PLATINUM, in our Communities of Inquiry, we work within university systems in which teaching and learning practices in mathematics have been embedded over decades, centuries or even millennia. Changing these is hugely demanding and very difficult for many reasons, some articulated above. To work within these practices, it is impossible not to align with the ways in which teaching and learning practices are shared and accepted. However, we do not have to do this uncritically. This is where inquiry comes in. By changing the idea of Community of Practice to Community of Inquiry, we start to open up possibilities. We can make small changes – for example, ask students more open questions, offer them an inquiry-based task and discuss it with them, start some occasional small-group work in which inquiry activity becomes the norm. It is through such efforts for small changes towards inquiry approaches that we engage with critical alignment—we align critically—and we start to appreciate elements of inquiry-based practice and to engage with its demands.

The three-layer model of inquiry has been used as a framework to structure our work in PLATINUM and to describe the results according to the intellectual outputs (IOs). The PLATINUM project has worked with six IOs as agreed within the Erasmus+ programme. These are as follows:

IO1 focuses on the conceptual underpinnings of the project, providing a framework for dealing with *inquiry*, *inquiry communities*, *inquiry-based learning and teaching*, and *critical alignment.* This chapter (2) explains conceptual underpinnings in the PLATINUM Project and sets the scene for the contents and structure of the following chapter.

IO2 focuses on *Communities of Inquiry.* The whole PLATINUM project as well as each partner group is committed to establishing a *community of inquiry* between its participants. Chapter 3 explores some of the underpinnings of the concept of community of inquiry. Through the idea of a *spiderchart*, it explores concepts and constructs that can contribute with differing degrees to a CoI. The central features of a CoI are the ways we work together to promote inquiry-based practices for the benefit of our students' learning and understanding in mathematics. Discussion in our CoI around the constructs of the spidercharts can help the community to develop. We do not need all to think in exactly the same ways—inquiry is about exploring possibilities and learning from our exploration whether in mathematics or in mathematics teaching; it is not about everyone working or thinking identically. Inquiry-based practice can look different for different people, but the principles of inquiry can be shared for mutual advantage and individual development.

IO3 focuses on *Inquiry-Based Tasks and Teaching Units* that are designed for use by teachers with their students. Chapter 6 addresses this design and the examples that have emerged. In each of our partner groups we have engaged with practices of teaching and learning: giving lectures, tutorials, seminars, small group work; working with our students on different courses, setting examinations and marking them, awarding grades. This is our practice. Inquiry can enter into any of these elements of practice in many different ways. One of the tasks we have undertaken is that of designing inquiry-based tasks and teaching units for our students' engagement. These look different depending on the designer, the students, the mathematical topic for design, the nature of the teaching/learning event (lecture, tutorial etc.) and the desired learning outcomes. We are in the process of developing a compendium of tasks and teaching units with contributions from all partners. The activity in each partner group is specific to that group, relating to local context and culture and to ways of being and doing. We have no intention to make the groups look all the same. The case study chapters in Part 3 of this book present activity, perspectives and the learning and development in the different partner communities. One important focus here is the identified needs of students. Although every student is different with their own particular characteristics and needs, some students have well defined needs, either physical or psychological (e.g., sight or hearing; dyslexia or autism). Two of our partner universities have special centres for the support of students with identified learning needs. Colleagues in these centres have guided the rest of us in making provision for these students. Chapter 4 focuses specifically on provision for identified needs and Chapter 6 includes identified needs in relation to the tasks and teaching units.

IO4 focuses on *Professional Development* for new and experienced teachers/lecturers who are interested in developing inquiry-based learning and teaching in mathematics. Chapter 7 addresses such professional development by offering methods and materials for professional use. We are well aware, in PLATINUM, that inquiry-based practice is itself an important source of professional development as can be seen in the chapters in Part 3. As we have worked together during nearly three years, it is fair to say that we have all developed in differing ways, depending on our starting points and the activities in which we have engaged. We are all more knowledgeable in what inquiry means for us, in *what we can do* to engage with inquiry, and the *differing ways* in which we can engage. If new colleagues join us, we can draw them into our communities and they can learn through working alongside others with critical alignment.

However, it is sometime necessary, or requested, that we offer some professional development workshops or seminars in inquiry-based practice for the benefit of others who wish to hear about our experiences and associated practices. For this purpose, we experimented with three workshops, each in one of our partner settings. The workshops had some activities in common and some specific to the setting. What we learned from these forms the basis of Chapter 7 in which we address *professional development in* PLATINUM and provide ideas for workshops etc to embrace and inform interested colleagues.

IO5 focuses on *Mathematical Modelling* addressing the design and use of inquiry-based tasks that relate to the world around us. Chapter 8 addresses this design and the examples that have emerged.

IO6 focuses on *Evaluation.* This includes both evaluation of our use of inquiry-based materials with students in our Partner locations as well as the wider evaluation of the project as a whole. Chapter 9 addresses issues in evaluation.

We highlight some of these interrelations among IO's according to the chapters.

Relations between IO1 and representative local cases (IO2). All partners contributed to the development of a generic framework for inquiry-based teaching of mathematics at the university level and in different contexts of higher education. It is worth noting that Chapter 15 reflects a long history of implementing the model, while other chapters express differing levels of experience in inquiry-based practice.

Relations between IO2, IO3, and IO5. The development of local university mathematics lecturers' communities of inquiry in which the university lecturers have explored inquiry-based teaching and learning and reflected on the effects in the design and implementation of teaching units. For example, Chapter 18 reports inquiry-based teaching approaches including expertise from industry and presentation of worked examples of realistic mathematical models. Also, see Chapters 13 and 17 written by mathematicians and experts in diversity (identified needs). These show productive results in contributing to IBME elements typically in large courses (including mathematics and statistics for economists).

Relations between IO4 and IO3 in the development of a professional development programme. Here, not only theoretical frameworks of professional development, but also the results and experience of the cycles concerning the design of the tasks or units in IO3 were taken into account. The development of collections of teaching units at the local level promoted mathematical conceptual learning through an inquiry approach. The nuances brought by each implementation in the specific regular courses in different Bachelor degrees (biomedical science, mathematics, computer engineering, etc.) at several universities have enriched PLATINUM's global approach to professional development of mathematics lecturers (see Chapters 11, 14, and 16). The contributions of these chapters allow us to give answers from PLATINUM to different questions such as: What does inquiry-based mathematics education means and why do we prefer inquiry-oriented education for professional development courses? What went well (or even above expectation), what was less successful, and what kind of obstacles were encountered? What ideas came up to improve the course in terms of contents, learning outcomes, and pedagogy (in particular opportunities for student inquiry)?

Relationship of the IO6 with other IO's, Developing the third layer of the CoI. PLATINUM project has made an effort to transmit the evaluation experience of the different inquiry communities (See Chapter 7). Some concepts related to IBME such as (1) evaluation of conceptual learning and teaching of mathematics; (2) monitoring students' engagement in IBME; (3) evaluation of the teaching practice by the CoI; and (4) evaluation of experiences about professional development of university mathematics lecturers. Each partner has contributed to different layers of the model. We have been able to share evaluation commonalities, but also characterise differences. The common aspect of the cases presented is the commitment in the CoI and the subsequent commitment to develop research on Inquiry based teaching and learning in light of what was learned in the process. This, as we have seen above, is the third layer of the CoI and the one that needs to be developed in time.

2.6. Discussion and Conclusions

Our writing. above, addresses not only the theoretical basis of our chapter, our inquiry approach towards mathematics learning and teaching at university level, but also the reasons why such an approach could be needed and should be considered. Although, in PLATINUM, our Communities of Inquiry bring together colleagues from different traditions within Europe, the needs for improvement of mathematics teaching and learning at Higher Education level are largely commonly understood and agreed. Despite national differences, we all find students who may not experience mathematics in the ways we would like to see, as articulated above. Although this might be widely recognised in most environments, the developmental solutions are not widely understood or agreed. However, through conceptualisation of the PLATINUM project, our eight partners came to agreement on many of the issues involved and a commitment to addressing them. IBME provided a possible basis for our exploration, which included our theoretical model and its characterisation as a framework for our developmental activity.

One, very obvious difference, in our PLATINUM partnership, was that some of us are mathematicians (doing research in mathematics) and some are mathematics educators (doing research in mathematics education and, in some cases, in educating new teachers). A small number are both. All of us are teachers. In this book we emphasise our teaching and its development, the learning that we have experienced through exploring new forms of practice and considering carefully the outcomes and issues. This is what we mean by 'walking the walk'. The case studies in Part 3 offer insights to the developmental processes in which we have engaged. These insights have brought us closer together as partners, exploring what inquiry can mean in our teaching and for our students. Together we have designed tasks and used them with students, in our teaching. Researching the nature of this process of design-use-feedback has enabled us to gain insights leading to new knowledge in practice. The other chapters in this book address the various dimensions of our work, reflected in the intellectual outputs (IOs).

The *essence* of the PLATINUM project lies in its interpretation of inquiry-based learning and teaching in mathematics. This was not pre-given. Working from the three-layer model, our framework for activity developed throughout the project. At workshops in each of our countries, we shared our thinking and our associated activity. Design of inquiry-based mathematical tasks became central to that activity and provided a rich common ground for discussion and experimentation. The concept of Inquiry Community—working together collaboratively, both in and across our partner groups—brought with it fellowship and understanding, a willingness to learn with and from each other in a variety of ways. Other chapters, in Parts 1 and 2 of this book, provide details of this work and its outcomes. They are written by teams of authors from across our partner groups. They bring together perspectives from across the project, unifying partner perspectives in identifying common aims and practices, and providing examples. Part 3 consists of eight case studies, each one presenting key elements of the activity and development of one partner group. Here we see diversity, both in terms of the starting points for each group, and also the developmental directions their activity and learning have taken.

As you read this book, we hope you will enter into our activity, its modes of inquiry, and the issues we have addressed. For us, the new insights we have gained provide a rich basis for further activity in applying and understanding elements of inquiry-based progress in learning and teaching to provide, we hope, better learning experiences of mathematics for our students.

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